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NEW FINITE DIFFERENCE SCHEMES FOR WAVE EQUATION MIGRATION

Róbert MÄRLE*

The paper describes some new difference schemes for wave equation migration. The favourable properties of these schemes from the point of view of the 3D problem will be pointed out.

d: wave equation migration, finite differences, 3D seismics

1. Introduction

In the last few years several publications have been devoted to wave equation migration. The problem of velocity inhomogeneities and of the appropriate boundary conditions have been dealt with in several places. Less attention has been given to the method of finite differences compared with the vast literature on frequency-domain migration. The initial successes achieved with ELGI's finite-difference migration programs and the new tasks which have arisen in 3D processing motivated our undertaking this study.

The research has centred around two tasks: ways to improve the treatment of greater dips, and novel possibilities for the execution of 3D migration in one step.

The first part of the paper will describe a negative result. The scheme to approximate the original equation will prove to be absolutely stable but not consistent. It will also be shown, heuristically, why we have not succeeded in finding a consistent scheme.

In the second part of the paper difference schemes will be introduced which can be used for the solution of the 3D migration in a single step.

2. Approximation of the full equation

CLAERBOUT [1976] derived the wave-equation

$$u_{xx} + u_{zz} + \frac{2}{v} u_{zt} = 0 \tag{1}$$

* Eötvös Loránd Geophysical Institute of Hungary, POB 35, Budapest, H-1440
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Upon neglecting the u_{zz} term, we get the so-called "15° equation", which is very similar to the equation of heat conductivity. Claerbout introduced an absolutely stable, implicit difference scheme. While one has to stick to the requirement of absolute stability (since the ratio of the lattice constants (Δx, Δt, Δz) can be widely different), the requirement of the implicit difference schemes can be dropped. Since the 15° and 45° equations do not properly treat the very steep dips, it is reasonable to assume that if we could solve the complete equation, this would provide a better migration procedure for the very large dips.

For the equation of heat conduction quite a few difference schemes have been proposed, including explicit and absolutely stable ones, but even these are only conditionally consistent. To put it plainly, stability means that the numerical errors cannot accumulate during the solution so as to make the result quite useless, while consistency means that we are really approximating the same equation from which we have started out. Exact mathematical definitions can be found, for example, in the monograph of RICHTMYER and MORTON [1967]. Consider the equation of heat conduction

$$u_t = u_{xx} \tag{2}$$

The so-called Du-Fort—Frankel difference scheme will be:

$$\frac{u_k^{n+1} - u_k^{n-1}}{2\Delta t} = \frac{u_{k-1}^n - u_k^{n+1} - u_k^{n-1} + u_{k+1}^n}{(\Delta x)^2} \tag{3}$$

that is

$$u_k^{n+1} + au_k^{n+1} = au_{k-1}^n + au_{k+1}^n - au_k^{n-1} - u_k^{n-1}$$

$$(1+a)u_k^{n+1} = a(u_{k-1}^n + u_{k+1}^n) - (1-a)u_k^{n-1} \tag{4}$$

with $$a = \frac{2\Delta t}{(\Delta x)^2}$$

If we check the stability of this difference scheme by Neumann's method [cf. RICHTMYER and MORTON 1967], it turns out that it is stable independently of the value of a (if $a > 0$, but this obviously holds). Indeed, the amplification factor ξ of the Neumann method satisfies the equation

$$(1+a)\xi^2 - 2a\cos\varphi \cdot \xi - (1-a) = 0$$

from where

$$\xi_{1,2} = \frac{2a\cos\varphi \pm \sqrt{4a^2\cos^2\varphi + 4 - 4a^2}}{2(1+a)}$$

If the discriminant is negative, ξ_1 and ξ_2 are complex conjugates of each other, i.e. their moduli are equal, and

$$|\xi_{1,2}|^2 = \frac{1-a}{1+a} < 1 \qquad a>0$$

If the discriminant is positive, we can also easily prove that the moduli of ξ_1 and ξ_2 are less than 1.

The only problem with the difference scheme (3) is its consistency with the equation

$$u_t = u_{xx} - \frac{(\Delta t)^2}{(\Delta x)^2} u_{tt} \tag{5}$$

rather than with the original equation. If $\frac{\Delta t}{\Delta x}$ is sufficiently small, this does not cause serious errors.

The importance of consistency lies in the fact that the result obtained from the finite difference solution approximates the solution to that equation which is consistent with the difference scheme [Theorem of LAX, see RICHTMYER and MORTON op.cit.]. The consistency of a difference scheme can be studied by means of Taylor series. The absolute stability and conditional consistency of the Du-Fort—Frankel scheme are both due to the fact that in the approximation of u_{xx} the term $2u_k^n$ was substituted by the sum $(u_k^{n+1} + u_k^{n-1})$. Had they left the original terms as $2u_k^n$ the scheme would be consistent but by no means absolutely stable.

For some considerable time, the author has been attempting to find an absolutely stable difference scheme consistent with Eq. (1). Although these attempts have not been successful so far, the author has succeeded in finding an absolutely stable approximation (hereinafter we shall use the notation $u(n\Delta z, k\Delta t, l\Delta x) = u_{k,l}^n$):

$$\begin{aligned}
&\frac{1}{(\Delta z)^2}(u_{k,l}^{n+1} - u_{k,l+1}^{n} - u_{k,l-1}^{n} + u_{k,l}^{n-1}) + \\
&\frac{1}{2v\Delta z\Delta t}(u_{k+1,l}^{n+1} - u_{k-1,l}^{n+1} - u_{k+1,l}^{n-1} + u_{k-1,l}^{n-1}) = \\
&\quad - \frac{1}{(\Delta x)^2}((u_{k+1,l-1}^{n} + u_{k-1,l-1}^{n})/2 - \\
&\quad - (u_{k-1,l}^{n-1} + u_{k-1,l}^{n+1} + u_{k+1,l}^{n-1} + u_{k+1,l}^{n+1})/2 + \\
&\quad + (u_{k+1,l+1}^{n} + u_{k-1,l+1}^{n})/2)
\end{aligned} \tag{6}$$

It will be shown later that this scheme is absolutely stable; the consistency condition, however, is not met, for Eq. (6) is consistent with the equation

$$u_{zz} - \left(\frac{\Delta x}{\Delta z}\right)^2 u_{xx} + \frac{2}{v} u_{zt} = -u_{xx} + \left(\frac{\Delta z}{\Delta x}\right)^2 u_{zz} + \left(\frac{\Delta t}{\Delta x}\right)^2 u_{tt} \tag{7}$$

The applicability of this approximation for migration purposes has been checked by model experiments. The results obtained have generally met the expectations, though they have also revealed a few surprising features. The experiments were carried out on a mathematical model section, the obtained results were compared with that of a conventional 45° migration program. The model consists of dipping reflecting surfaces whose migrated picture can easily be constructed. On the whole, the 45° migration has put these reflecting surface elements in their proper place, with characteristic long "smiles" at their ends. On the other hand, scheme (6) has not put the surfaces in their proper place, they have only been displaced about half-way; their shortening however is exactly the same as theoretically expected. The lesson from the experiment is, of course, that this difference scheme cannot be applied for migration, though it should be added that when we selected Δz to be two times less than in the 45° equation, the result became much better and the time of computation was only about half that of the 45° migration. This latter phenomenon can easily be understood if we recall that the error of the consistency depends on Δz in an almost random manner, which makes the difference scheme practically inapplicable. The stability of the difference scheme can be established as follows. Multiplying Eq. (6) by $(\Delta z)^2$ and letting

$$a = \frac{(\Delta z)^2}{(\Delta x)^2} \qquad b = \frac{\Delta z}{2v\Delta t} \qquad \text{we obtain}$$

$$\begin{aligned} & u^{n+1}_{k,l} - u^n_{k,l+1} - u^n_{k,l-1} + u^{n-1}_{k,l} + b(u^{n+1}_{k+1,l} - u^{n+1}_{k-1,l} - u^{n-1}_{k+1,l} + u^{n-1}_{k-1,l}) = \\ & -a((u^n_{k+1,l-1} + u^n_{k-1,l-1})/2 - (u^{n-1}_{k-1,l} + u^{n-1}_{k+1,l} + u^{n+1}_{k-1,l} + u^{n+1}_{k+1,l})/2 + \\ & + (u^n_{k+1,l+1} + u^n_{k-1,l+1})/2) \end{aligned} \tag{8}$$

Rearranging the equation with respect to the index n

$$\begin{aligned} & u^{n+1}_{k,l} + a({}^{n+1}_{k-1,l} + u^{n+1}_{k+1,l})/2 + b(u^{n+1}_{k+1,l} - u^{n+1}_{k-1,l}) = \\ & = u^n_{k,l+1} + u^n_{k,l-1} - a(u^n_{k+1,l-1} + u^n_{k-1,l-1} + u^n_{k-1,l+1} + u^n_{k+1,l+1})/2 + \\ & + b(u^{n-1}_{k+1,l} - u^{n-1}_{k-1,l}) - a(u^{n-1}_{k-1,l} + u^{n-1}_{k+1,l})/2 - u^{n-1}_{k,l} \end{aligned} \tag{9}$$

and we get, by Neumann's method, the following equation for the amplification factor:

$$(1 - a\cos\omega\Delta t + 2ib\sin\omega\Delta t)\xi^2 = (2 - 2a\cos\omega\Delta t)\cos k_x\Delta x\xi - 1 - a\cos\omega\Delta t + 2ib\sin\omega\Delta t \tag{10}$$

With the further notations $C = 1 - a\cos\omega\Delta t$; $D = 2ib\sin\omega\Delta t$ we have

$$(C + iD)\xi^2 - 2C\cos k_x\Delta x\xi + C - iD = 0 \tag{11}$$

that is

$$\xi_{1,2} = \frac{2C\cos k_x\Delta x \pm \sqrt{4C^2\cos^2 k_x\Delta x - 4(C^2 + D^2)}}{2(C + iD)} = $$
$$= \frac{C\cos k_x\Delta x \pm i\sqrt{C^2 + D^2 - C^2\cos^2 k_x\Delta x}}{C + iD}$$

Upon multiplying both the numerator and denominator by the complex conjugate of the latter, we get

$$|\xi_{1,2}|^2 = \frac{C^2 + D^2}{C^2 + D^2} = 1 \tag{12}$$

that is, the difference scheme is stable.

Let us now consider a general scheme, based on the approximations

$$\begin{aligned}\hat{u}_{zz} = {} & a\,(u^{n-1}_{k-1,\,l-1} - 2u^{n}_{k-1,\,l-1} + u^{n+1}_{k-1,\,l-1}) + a_2\,(u^{n-1}_{k-1,\,l} - 2u^{n}_{k-1,\,l} + u^{n+1}_{k-1,\,l}) + \\ & + a_3(u^{n-1}_{k-1,l+1} - 2u^{n}_{k-1,l+1} + u^{n+1}_{k-1,l+1}) + a_4(u^{n-1}_{k,l-1} - 2u^{n}_{k,l-1} + u^{n+1}_{k,l-1}) + \\ & + a_5(u^{n-1}_{k,l} - 2u^{n}_{k,l} + u^{n+1}_{k,l}) + a_6(u^{n-1}_{k,l+1} - 2u^{n}_{k,l+1} + u^{n+1}_{k,l+1}) + \\ & + a_7(u^{n-1}_{k+1,l-1} - 2u^{n}_{k+1,l-1} + u^{n+1}_{k+1,l-1}) + a_8(u^{n-1}_{k+1,l} - 2u^{n}_{k+1,l} + u^{n+1}_{k+1,l}) + \\ & + a_9(u^{n-1}_{k+1,l+1} - 2u^{n}_{k+1,l-1} + u^{n+1}_{k+1,l+1})\end{aligned} \tag{13}$$

with $\sum_{i=1}^{9} a_i = 1$

$$\begin{aligned}\hat{u}_{xx} = {} & b_1(u^{n-1}_{k-1,l-1} - 2u^{n-1}_{k-1,l} + u^{n-1}_{k-1,l+1}) + b_2(u^{n-1}_{k,l-1} - 2u^{n-1}_{k,l} + u^{n-1}_{k,l+1}) + \\ & + b_3(u^{n-1}_{k+1,l-1} - 2u^{n-1}_{k+1,l} + u^{n-1}_{k+1,l+1}) + b_4(u^{n}_{k-1,l-1} - 2u^{n}_{k-1,l} + u^{n}_{k-1,l+1}) + \\ & + b_5(u^{n}_{k,l-1} - 2u^{n}_{k,l} + u^{n}_{k,l+1}) + b_6(u^{n}_{k+1,l-1} - 2u^{n}_{k+1,l} + u^{n}_{k+1,l+1}) + \\ & + b_7(u^{n+1}_{k-1,l-1} - 2u^{n+1}_{k-1,l} + u^{n}_{k-1,l+1}) + b_8(u^{n+1}_{k,l-1} - 2u^{n+1}_{k,l} + u^{n+1}_{k,l+1}) + \\ & + b_9(u^{n+1}_{k+1,l-1} - 2u^{n+1}_{k+1,l} + u^{n+1}_{k+1,l+1})\end{aligned} \tag{14}$$

with $\sum_{i=1}^{9} b_i = \frac{(\Delta z)^2}{(\Delta x)^2}$

and letting

$$\begin{aligned}\hat{u}_{zt} = {} & C_1(u^{n+1}_{k+1,\,l-1} - u^{n+1}_{k-1,\,l-1} - u^{n-1}_{k+1,\,l-1} + u^{n-1}_{k-1,\,l-1}) + \\ & + C_2(u^{n+1}_{k+1,\,l} - u^{n+1}_{k-1,\,l} - u^{n-1}_{k+1,\,l} + u^{n-1}_{k-1,\,l}) + \\ & + C_3(u^{n-1}_{k+1,\,l+1} - u^{n+1}_{k-1,\,l+1} - u^{n-1}_{k+1,\,l+1} + u^{n-1}_{k-1,\,l})\end{aligned} \tag{15}$$

with

$$C_1 + C_2 + C_3 = \frac{1}{2v\Delta z\Delta t}$$

This difference equation

$$\hat{u}_{zz} + \hat{u}_{xx} + \hat{u}_{zt} = 0$$

is consistent with the original full equation (1).

The difference scheme is general, if we restrict ourselves to the 27 lattice points around the point $u^n_{k,\,l}$. Let us now search for an absolutely stable scheme among these schemes. If we arrange the above equation with respect to the index n, the following terms will belong to index $(n+1)$:

$$\begin{aligned}& a_1 u^{n+1}_{k-1,\,l-1} + a_2 u^{n+1}_{k-1,\,l} + a_3 u^{n+1}_{k-1,\,l+1} + a_4 u^{n+1}_{k,\,l-1} + \\ & + a_5 u^{n+1}_{k,\,l} + a_6 u^{n+1}_{k,\,l+1} + a_7 u^{n+1}_{k+1,\,l-1} + a_8 u^{n+1}_{k+1,\,l} + \\ & + a_9 u^{n+1}_{k+1,\,l+1} + b_7(u^{n+1}_{k-1,\,l-1} - 2u^{n+1}_{k-1,\,l} + u^{n+1}_{k-1,\,l+1}) + \\ & + b_8(u^{n+1}_{k,\,l-1} - 2u^{n+1}_{k,\,l} + u^{n+1}_{k,\,l+1}) + b_9(u^{n+1}_{k+1,\,l-1} - 2u^{n+1}_{k+1,\,l} + u^{n+1}_{k+1,\,l+1}) + \\ & + C_1(u^{n+1}_{k+1,\,l-1} - u^{n-1}_{k-1,\,l-1}) + C_2(u^{n+1}_{k+1,\,l} - u^{n+1}_{k-1,\,l}) + \\ & + C_3(u^{n+1}_{k+1,\,l+1} - u^{n+1}_{k-1,\,l+1})\end{aligned} \tag{16}$$

The same terms appear for $(n-1)$, only the signs of the c_i-s will be opposite and b_1, b_2, b_3 will figure. For the index n the c_i-s will be missing, b_4, b_5, b_6 will figure and the a_i – s will appear with coefficient (-2).

By Neumann's method, the amplification factor is the root of the second-order equation

$$(A + B_1 + C)\xi^2 - (2A - B_2)\xi + A + B_3 - C = 0 \tag{17}$$

If we choose the a_i – s symmetrically to a_5, the number A will be real. The number C is pure imaginary. Two cases should be distinguished: either $B_2 = 0$ and $B_1 = B_3$, or $B_2 = B_1 + B_3 = 2\,B_1$. In the first case the equation for the amplification factor assumes the form

$$(A+B+iF)\xi^2 - 2A\xi + (A+B-iF) = 0 \tag{18}$$

from where

$$\xi_{1,2} = \frac{2A \pm \sqrt{4A^2 - 4(A^2+B^2+2AB+F^2)}}{2(A+B+iF)} \tag{19}$$

In this case we shall have $|\xi| > 1$ for certain values of k_x, k_z, ω since the numbers A, B, F are functions of k_x, k_z, ω that is, the difference scheme is not stable for these values.

In the second case the equation becomes

$$(A+B_1+iF)\xi^2 - (2A-2B_1)\xi + (A+B_1-iF) = 0 \tag{20}$$

that is,

$$\xi_{1,2} = \frac{A-B_1 \pm \sqrt{(A-B_1)^2 - (A+B_1)^2 + F^2}}{A+B+iF} \tag{21}$$

The same situation is encountered as with Eq. (19), that is the difference scheme is not stable in either of the cases. We have not proved, of course, that there exists no absolutely stable difference scheme which would be consistent with the full equation: however, if it exists, its construction will certainly not be an easy task (and it will become more difficult with the incorporation of more lattice points).

3. Approximation of the 15° and 45° equations

Claerbout and Johnson [1971] proposed the following approximation in their solution of the 15° equation:

$$u_{k+1}^{n+1} - u_k^{n+1} - u_{k+1}^n + u_k^n = a\mathrm{T}(u_{k+1}^{n+1} + u_k^{n+1} + u_{k+1}^n + u_k^n) \tag{22}$$

where $\mathrm{T}u = u_{l-1} - 2u_l + u_{l+1}$ is the operator of the second difference with respect to x, and

$$a = \frac{v\Delta z \cdot \Delta t}{4(\Delta x)^2}.$$

This scheme is absolute stable, and consistent with the 15° equation. It has been widely used for 2D migration and its properties are well known. If we try to use it, however, for 3D migration, we face the obstacle that the arising system of equations will have a block-tridiagonal, rather than tridiagonal, matrix which causes a substantial increase in computation time. It would be very important to find new absolutely stable and consistent difference schemes for which there are corresponding linear systems with diagonal matrices. A possible difference scheme arises from the equation

$$u_{k+1}^{n+1} - u_{k-1}^{n+1} - u_{k+1}^{n} + u_{k-1}^{n} = a\mathrm{T}(u_k^{n+1} + u_k^n) \tag{23}$$

Let us first determine the amplification factor:

$$u_{k+1}^{n+1} - a\mathrm{T}u_k^{n+1} - u_{k-1}^{n+1} = u_{k+1}^{n} + a\mathrm{T}u_k^n - u_{k-1}^{n} \tag{24}$$

$$u_k^{n+1}(-a\mathrm{T} + 2i\sin\omega\Delta t) = u_k^n(a\mathrm{T} + 2i\sin\omega\Delta t) \tag{25}$$

$$u_k^{n+1} = \frac{A+iB}{-A+iB}u_k^n \tag{26}$$

$$\xi = \frac{A+iB}{-A+iB} \qquad |\xi|^2 = \frac{A^2+B^2}{A^2+B^2} = 1 \tag{27}$$

which proves the stability of the scheme.

In 3D migration, derivatives with respect to y must also be taken into account, i.e. the scheme should be correspondingly modified. In the actual execution of the 3D migration by means of this scheme, we have to simultaneously store 6 vectors as against the 4 vectors needed in Claerbout's scheme. This apparent drawback is counterbalanced by the fact that if we were to carry out 3D migration in a single step by Claerbout's method, the number of operations would increase proportionally to the number of parallel profiles. For example, for an areal survey consisting of 78 × 85 CDP traces, the migration would require 78 times as much time. One way to circumvent this difficulty is the two-step migration [Ristow and Houba 1979], which, however, causes further inaccuracies in the solution. An additional advantage of the one-step migration is that it saves a great deal of data transfers and requires fewer operations. The 45° equation can also be approximated by difference schemes of the same properties as scheme (23).

A possible choice is:

$$\begin{aligned}\frac{\mathrm{T}}{a}(u_{k+1}^{n+1} - u_{k+1}^{n} + u_k^{n+1} - u_k^n) - \frac{1}{b}(u_{k+2}^{n+1} - u_{k+2}^{n} +\\ u_{k+1}^{n} - u_{k+1}^{n+1} - u_k^{n+1} + u_k^n + u_{k-1}^{n+1} - u_{k-1}^{n}) =\\ = \frac{\mathrm{T}}{c}(u_{k+1}^{n} - u_k^n + u_{k+1}^{n+1} - u_k^{n+1})\end{aligned} \tag{28}$$

where

$$a^{-1} = \frac{1}{(\Delta x)^2 \Delta z}, \; b^{-1} = \frac{4}{v^2 \Delta x (\Delta t)^2}, \; c^{-1} = \frac{2}{v(\Delta x)^2 \Delta t} \tag{29}$$

It can be shown that the difference scheme (28) is consistent with the so-called 45° equation:

$$u_{xxz} = \frac{4}{v^2} u_{ztt} + \frac{2}{v} u_{xxt} \tag{30}$$

The proof of consistency has been omitted as it is quite lengthy and does not require any original thoughts beyond the usual manipulations. This difference scheme also has the drawback that more data should be stored, and the same advantageous property: during solution the system of equations decomposes into independent subsystems.

Stability can be shown as follows. Rearranging Eq. (28) with respect to index k we get

$$\begin{aligned} &\frac{T}{a}(u_{k+1}^{n+1} + u_k^{n+1}) - \frac{1}{b}(u_{k+2}^{n+1} - u_{k+1}^{n+1} - u_k^{n+1} + u_{k-1}^{n+1}) - \\ &\frac{T}{c}(u_{k+1}^{n+1} - u_k^{n+1}) = \frac{T}{a}(u_{k+1}^n + u_k^n) - \\ &- \frac{1}{b}(u_{k+2}^n - u_{k+1}^n - u_k^n + u_{k-1}^n) + \\ &+ \frac{T}{c}(u_{k+1}^n - u_k^n) \end{aligned} \tag{31}$$

By Neumann's method, the amplification factor ξ will assume again the form

$$\xi = \frac{A - iB}{A + iB} \tag{32}$$

that is, $|\xi| = 1$, which proves stability.

It should finally be noted that these new difference schemes have not yet been checked on actual seismic data: their applicability, advantages and disadvantages, will only be proved by practice.

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ÚJABB DIFFERENCIA-SÉMÁK A HULLÁMEGYENLET MIGRÁCIÓHOZ

MÄRLE Róbert

A cikk olyan differencia-sémákat ír le, amelyeket eddig nem használtak a hullámegyenlet migrációhoz. Ismerteti az új sémák tulajdonságait, különös tekintettel azokra, amelyek a 3 D migráció szempontjából jelentősek.

НОВЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ СХЕМЫ ДЛЯ МИГРАЦИИ ПО ВОЛНОВОМУ УРАВНЕНИЮ

Р. МЭРЛЕ

В работе дается описание дифференциальных схем, которые до сих пор не были использованы для миграции по волновому уравнению. Обсуждаются особенности новых схем, причем особое внимание придается тем, которые считаются важными с точки зрения пространственной миграции.