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# AN ANALYSIS OF THE AXIAL FLOW COMPRESSOR STABILITY USING THE BIFURCATION THEORY

## INTRODUCTION

Axial flow compressors are designed to operate in steady axisimetric flow. If the mass flow is decreased, the pressure rise increases but, unfortunately a critical value is reached, beyond steady flow is no longer stable; a small change in the flow may be enough to push one into the unstable region. It is important to know the nature of the flow that develops in this situation, as the different flows can have quite different consequences.

In this paper it will be shown how bifurcation theory can be used to characterise each type of flow and locate the regions of parameter space where such a flow is stable. The flow regimes are: axisymetric steady, nonaxisymetric steady, axisymetric time dependent and non-axisymetric time dependent and are known respectively as design flow, rotating stall, deep surge and classic surge.

It is possible to reduce the Moore-Greitzer model for compressor instability to a set of three ordinary differential equations using a one-mode truncation in bifurcation theory. The flow entering the inlet is uniform and the duct is straight (radial variations are negligible);the flow is assumed to be incompressible and irrotational (the velocity potential of any disturbance which develops in the inlet satisfies Laplace's equation). The unsteady flow is 2D in axial distance and circumferential angle and the general solution is:

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n(t) e^{in\theta} \left\{ e^{n(\eta + l_I)} + e^{-n(\eta + l_I)} \right\} + c.c. \tag{1}$$

where $\tilde{\phi}$ is the potential function for the disturbance in the inlet. The independent variables are time (nondimensionalised with the time for the wheel to rotate one radian), angular position and axial distance (nondimensionalised with the wheel radius). The disturbance that reaches the compressor face is assumed to move straight through the compressor to the plenum and as a result the only independent variables are and time, t. The rate of change of mass flow through the compressor can be related to the pressure rise in the following way:

$$\frac{l_c}{S}\Phi'(t) = -\Psi(t) + \frac{1}{2\pi}\int_0^{2\pi}\Psi_c(\phi)\,d\phi \tag{2}$$

where Φ represents the angle averaged axial mass flow and Ψ represents the angle averaged pressure rise, with the prime denoting differentiation with respect to t. The mass flow coefficient and pressure rise coeffficient have been rescaled and shifted so that:

$$\Phi = \frac{\text{mass flow coefficient}}{W} - 1$$

$$\Psi = \frac{\text{pressure rise coefficient}}{H}.$$

Parameters W and H are scalling factors chosen so that the resulting performance curve is representative for all compressors at all speeds. Let $P_{max}$ be the maximum pressure rise coefficient and $M_{max}$ be the corresponding mass flow coefficient. Then:

$$W = \frac{M_{\max}}{2}$$

$$H = \frac{P_{\max} - \Psi_o}{2}.$$

The axial mass flow coefficient is:

$$\phi(\theta,t) = \Phi(t) + \tilde{\phi}_{,\eta|\eta=0}$$

where $\tilde{\phi}_{,\eta|\eta=0}$ is the axial flow disturbance.

The function $\Psi_c(\phi)$ in (2) represents the response of the compressor in steady axisymetric flow; it is a typically S-shaped curve (Moore-Greitzer):

$$\Psi_c(\phi) = \Psi_{co} + 1 + \frac{3}{2}\phi - \frac{1}{2}\phi \tag{3}$$

The parameter $\Psi_{co}$ reflects the number of stages in compressor, $l_c$ represents the inertial length of the compressor and the scalling parameter *S=H/W*.

When the angle variations are taken into account and the pressure rise is summed over each component of the engine, we arrive at the following expression which acts as the last boundary condition for Laplace's equation in the inlet:

$$\left[m\tilde{\phi}_{,t} + \frac{1}{a}\tilde{\phi}_{,\eta t}\right]_{|\eta=0} = -l_c\Phi'(t) - S\Psi(t) + \Psi_c(\phi) - \frac{1}{2a}\tilde{\phi}_{,\eta\theta|\eta=0} \tag{4}$$

where $\tilde{\phi},_{\eta|\eta=0}$ is the axial mass flow disturbance at the compressor face. Parameter m reflects the type of exit duct (*m=1* for abrupt expansion; *m=2* for the long channel); parameter *a* is associated with the time lag of the flow in between the rows of blades.

Assuming the process in the plenum polytropic, the mass continuity equation for the plenum chamber is:

$$l_c \Psi'(t) = \frac{1}{(2B)^2 S}\left[\Phi(t) - \Phi_T(\Psi)\right], \tag{5}$$

where $\Phi_T$ is the mass flow coefficient leaving the chamber. The pressure rise through the throttle is modelled by the simple parabolic relationship:

$$\Psi = \frac{1}{\gamma}(\Phi_T + 1)^2 \tag{6}$$

where $\gamma$ is the control parameter for the exit mass flow.

To simplify the parameter groupings we define a new time variable:

$$\xi = t\frac{S}{l_c}$$

Substituting this into (1),(2),(4),(5) and recalling the solution for the inlet flow field gives the "full model". The prime now denotes differentiation with respect to time.

$$\Phi'(\xi) = -\Psi(\xi) + \frac{1}{2\pi}\int_0^{2\pi}\Psi_c(\phi)\,d\theta \tag{7}$$

$$\Psi'(\xi) = \frac{1}{(2BS)^2}\left[\Phi(\xi) - \Phi_T(\Psi)\right] \tag{8}$$

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n(\xi)\, e^{in\theta}\left\{e^{n(n+l_I)} + e^{-n(n+l_I)}\right\} + c.c. \tag{9}$$

$$\frac{1}{l_c}\left[m\tilde{\phi},_\xi + \frac{1}{a}\tilde{\phi},_{\eta\xi}\right]\Bigg|_{\eta=0} = -\Phi'(\xi) - \Psi(\xi) + \Psi_c(\phi) - \frac{1}{2aS}\tilde{\phi},_{\eta\theta}\Bigg|_{\eta=0} \tag{10}$$

## ONE MODE TRUNCATION

This simplification has been added advantage that eight parameters of the full model can now be reduced to four. It is assumed that the second and higher modes have negligible amplitude:

$$\tilde{\phi} = a_1(\xi) e^{in\theta} \left\{ e^{n(n+l_I)} + e^{-n(n+l_I)} \right\} + c.c. \qquad (11)$$

Substituting into (10) gives:

$$\frac{1}{l_c}\left[ m\left(e^{l_I} + e^{-l_I}\right) + \frac{1}{a}\left(e^{l_I} - e^{-l_I}\right) \right] a_1'(\xi) =$$
$$= \left[ -\frac{i}{2aS} + \frac{3}{2}\left(1-\Phi^2\right)\left(e^{l_I} - e^{-l_I}\right) \right] a_1 - \frac{3}{2}\left(e^{l_I} - e^{-l_I}\right) a_1^2 \bar{a}_1$$

The complex coefficient $a_1$ can be written as:

$$a_1(\xi) = \frac{r(\xi)}{\left(e^{l_I} - e^{-l_I}\right)} e^{i\delta\xi} \qquad (12)$$

where $r$ is the amplitude and $\delta$ is the phase speed of an angular disturbance. Substituting this into the above and separating real and imaginary parts gives

$$\frac{r'(\xi)}{r} = \frac{\sigma}{2}\left(1-\Phi^2 - r^2\right); \quad \delta = -\frac{\sigma}{6aS}$$

where:

$$\sigma = \frac{3 l_c a \left(e^{l_I} - e^{-l_I}\right)}{m a \left(e^{l_I} - e^{-l_I}\right) + \left(e^{l_I} - e^{-l_I}\right)} \qquad (13)$$

The equation for $r$ can be further simplified by introducing $R=r^2$ . Equation (8) is unchanged and the integral in (7) can be solved explicitly. Equations (7)-(11) are thus reduced to the following set of three ordinary differential equations:

$$\Phi'(\xi) = -\Psi + \Psi_c(\Phi) - 3\Phi R \qquad (14)$$

$$\Psi'(\xi) = \frac{1}{\beta^2}\left(\Phi - \Phi_T(\Psi)\right) \qquad (15)$$

$$R'(\xi) = \sigma R \left(1 - \Phi^2 - R\right) \qquad (16)$$

where $\beta=2BS$. Both $\Psi$ and $R$ can assume only positive values; the former because of physical constrains and the latter because it is a squared quantity.
Equations (14)-(16) contain the essential dynamics of the physical problem. Using the bifurcation theory to define the boundaries for different flow regimes in the ($\beta,\gamma,\Psi_{co},\sigma$) parameter space is posiible to determine the possible solutions of this set of ordinary differential equations before solving them numerically.

The problem of locating the stationary points of (8)-(10) and examining their stability can be expressed as $\dot{x} = F(\mu, x)$ where $x \in \Re^3$ , $\mu \in \Re^4$

The stationary values $x_0 = \{x_{0i}\}$ satisfy $F(\mu , x_{0i}) = 0$ and the stability of each $x_{0i}$ is governed by the eigenvalues of the linear operator $L_\mu = D_x\ F(\mu , x_{0i})$.When no eigenvalue has a positive real part, the stationary point is stable to small perturbations; if one of this has real part positive, then the equilibrium point is unstable. As the parameter μ is varied, qualitative changes may occur in the dynamics. These changes are called bifurcations and the parameter values are called bifurcations values. We first find the simplest bifurcations of the equilibria and latter discuss the bifurcations of the periodic orbits. A qualitative picture where the branches of the equilibria are shown in *(x, μ)* space is a bifurcation diagram. A bifurcation set consists of the loci of the bifurcation points in $\mu$ space. An examination of (14)-(16) shows that there are two equilibrium values of *R* ,

$$\boldsymbol{R} = 0$$

$$\boldsymbol{R} = 1 - \Phi^2$$

The first of these demands $\Psi = \Psi_c$, which defines the axisymetric characteristic. For the second case

$$\Psi = \Psi_{co} + 1 - \frac{3}{2}\phi + \frac{5}{2}\phi^3 \tag{17}$$

which defines the rotating stall characteristic. Linerising (14)-(16) about an equilibrium point $(\Phi_e, \Psi_e, R_e)$ gives:

$$\Phi'(\xi) = (\Psi'_c(\Phi_e) - 3R_e)\Phi - \Psi - 3\Phi_e R$$

$$\Psi'(\xi) = \beta^{-2}\Phi - \beta^{-2}\Phi'_T(\Psi_e)\Psi$$

$$R'(\xi) = -2\sigma\Phi_e R_e \Phi + \sigma\left(1 - \Phi_e^2 - 2R_e\right)R$$

The plane defined by *R=0* is invariant since all flows starting on this plane will remain there forever. For all parameter values there is a fixed point $x_{01}$ on this plane at the intersection of the curves $\Psi_c(\Phi)$ and $\Psi(\Phi_T)$. Trajectories with *R=0* represent axisymetric flow and when fixed point is stable, it represents the steady design flow. At large values of γ the design flow represented by $x_{01}$ is stable. Decreasing γ corresponds to reducing the mass flow and causes $x_{01}$ to lose stability either at a transcritical or a pitchfork bifurcation point. We denote this bifurcation point as $\gamma_c$. At this stage we restrict our attention to the fixed points which are of interest physically, i.e., those which lie on or above the $R = 0$ plane. With $\Psi_{co} < 4$, two new fixed points, $x_{02}$ and $x_{03}$, results from the saddle node bifurcation at $\gamma = \gamma_s$. The latter is unstable (having a complex pair of eigenvalues and a positive real eingevalue) until it moves below the $R = 0$ plane at the transcritical bifurcation where $x_{01}$ loses stability. Thereafter it is no longer of physical interest. The stability of $x_{02}$ will be discussed momentarily. In the critical case, $\Psi_{co} = 4$, where $x_{01}$ loses stability at a pitcfork bifurcation point. In this case $x_{03}$ exists only for $R < 0$ and hence has no physical meaning. For $\Psi_{co} > 4$ the saddle node bifurcation lies below the $R = 0$ plane and $x_{01}$ now loses its stability at the transcritical bifurcation point with $x_{02}$ and $\gamma = \gamma_c$. When $x_{01}$ is stable, it represents the steady axisymmetric flow which is the design condition of the engine, and when $x_{02}$ is stable the flow condition modeled is steady rotating stall. The only occasions when $x_{03}$ is stable occur in the physically meaningless negative half/space of R. The possitions of the equilibrium points vary with γ, and the loci of the fixed points on the graph of pressure rise versus mass flow are called characteristics. The axismetric points lie on the cubic $\Psi_{co}(\Phi)$ axisymmetric characteristics). The non axisymmetric fixed points lie on another cubic, the rotating stall characteristic.

## CONCLUSIONS

The three ordinary differential equations were studied using the methods of bifurcation theory, which gave the boundaries in space parameter for each type of solution. The analysis shows the qualitative difference between deep surge (a purely axisymetric periodic orbit with actual trajectories showing some rotating stall initiated by the back ground distortion) and classic surge (associated with the Hopf bifurcation of the rotating stall point). The bifurcation analysis is of great value in the study of compressor instability. It provides a complete picture of the parametric effects in this simple model.

### REFERENCES

[1] BOSCOIANU Mircea, ZAGANESCU Florin, An Analyisis of Rotating Stall in Multistage Axial Compressors, 22nd International Congress of Aeonautical Sciences ICAS 2000.