Study of the Stabilization of Uncertain Nonlinear Systems Controlled by State Feedback

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Abstract: The control of a process by poles placement is one of the most used forms of feedback control. It allows not only to stabilize a process, but also to control its dynamic. Furthermore, the optimal controls with quadratic criteria of linear systems in fact lead to the pole placement. In this work, we present an approach to the stabilization of nonlinear systems in presence of uncertainties using poles placement by state feedback and the determination of attractors by diagonalization of the characteristic matrices linearized around operating points and using aggregation techniques.

Keywords: aggregation techniques; attractors; comparison systems; state feedback control; uncertain nonlinear

1 Introduction

The control of complex nonlinear process appears generally difficult, particularly in the case of ill-defined or imprecise models and when these processes are subject to unidentified noises or disturbances for which the only available information is the amplitudes of the uncertainties resulting in the definition of the model. A great number of works have been presented related to this problem [1-6]. For a nonlinear process in continuous time, whose evolution is described by a set of differential equations, the most commonly used model is represented in the state space.

However, starting from a set of given differential equations, several representations can be used and the choice of the model can affect the accuracy of the expected results.

In the presence of uncertainties in modeling, that increase the complexity of the stability study, it is not always possible to obtain a control law ensuring the stability of the process with respect to a chosen objective. It is then necessary to estimate the maximum deviation from this target, an operation which can be performed by determining an attractor corresponding to the vicinity of the purpose for which the local stability cannot be guaranteed [7-15].

Linear system stability study generally leads to necessary and sufficient conditions and doesn't depend, generally, on the system representation. The task is different for nonlinear systems with or without uncertainties, for which only sufficient conditions can be proposed; then the determination of their stability domains and attractors depends on the choice of both the description of the studied system and the used stability method [16-18].

Process control through poles placement is an usual feedback control used for linear systems [19]. It doesn't allow only to stabilize the studied process, but also imposes its dynamics. For nonlinear systems with uncertainties, the approach is more complex.

In the case of large scale systems, generally described in the state space, stability conditions are obtained, either directly for the whole system or separately for the various subsystems.

In this paper, the determination of the state feedback is based on a specific state space description of the linearized process and the determination of the attractor, when the process is submitted to uncertainties, is achieved by using aggregation techniques and the Borne-Gentina stability criteria, with the use of vector norms and of comparison systems [20-27].

The aim of this work is to present an approach to the study of stability of nonlinear systems and the estimation, by overvaluation, of the attractor. In Section 2, we propose an attractor determination method by diagonalization of the linearized characteristic matrix around an operating point when the control law is achieved by poles placement and by the use of the aggregation technique for stability study. The determination of attractor for a third order nonlinear complex system is presented, in Section 3, to illustrate the efficiency of the proposed approach.

2 Proposed Attractor Determination Method

In this section the poles placement is determined on a linearized model of the initial system without uncertainties.

2.1 Determination of State Feedback Gain *L*

Let us consider the system (S) described by

$$\dot{x}(t) = A(.)x(t) + B(.)u(t) + B'(.) \tag{1}$$

with $A \in R^{n\times n}$, $B \in R^n$, $x \in R^n$, $u \in R$ and $B' \in R^n$ characterizing the influence of uncertainties.

By linearization of the system (1) without uncertainties, around the operating point x_0, it comes the correspondent linearized model (2)

$$\dot{x}(t) = A(0)x(t) + B(0)u(t) \tag{2}$$

assumed to be controllable.

The state feedback control law of (2) is defined in the form

$$u(t) = -Lx(t) \tag{3}$$

such that

$$L = \begin{bmatrix} l_0 & l_1 & \dots & l_{n-1} \end{bmatrix}, \quad L \in R^n \tag{4}$$

Note P_c the matrix of change of base such that

$$x = P_c x_c \tag{5}$$

which enables to describe the linearized system (1) without uncertainties in the controllable canonical form

$$\dot{x}_c(t)=A_c x_c(t) + B_c u(t) \tag{6}$$

with x_c the new state vector of the process, $A_c = P_c^{-1}AP_c$ and $B_c = P_c^{-1}B$.

After substituing (3) in (1), it comes for the process without uncertainties

$$\dot{x}_c(t) = A_c x_c(t) - B_c L x_c(t) \tag{7}$$

or

$$\dot{x}_c = P_c^{-1}AP_c x_c + P_c^{-1}BLP_c x_c \tag{8}$$

then

$$\dot{x}_c(t) = H_c\, x_c(t) \tag{9}$$

with

$$H_c = A_c \;\text{-}B_c L_c \tag{10}$$

and

$$LP_c = L_c \tag{11}$$

such that

$$L_c = -\begin{bmatrix} l_{c_0} & l_{c_1} & \ldots\ldots l_{c_{n-1}} \end{bmatrix} \tag{12}$$

L_c is the state feedback gain in the controllable base in which , the matrices A_c and B_c are written in the canonical controllable form. The characteristic polynomial of matrix A , $\mathrm{P}_A(\lambda)$.

$$\mathrm{P}_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 \tag{13}$$

is invariant by change of base. Then, we have $\mathrm{P}_{A_c}(\lambda) = \mathrm{P}_A(\lambda)$.

The matrix A_c , being in the companion canonical form, we can easily calculate the characteristic polynomial of the closed loop system characteristic matrix, noted $\mathrm{P}_{H_c}(\lambda)$,

$$\mathrm{P}_{H_c}(\lambda) = \det(\lambda I - (A_c - B_c L_c)) \tag{14}$$

By the choice of L_c , we can impose the coefficients of the characteristic polynomial such that

$$\begin{aligned} \mathrm{P}_{H_c}(\lambda) &= \mathrm{P}_{A-BL}(\lambda) \\ &= \lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_2\lambda^2 + \alpha_1\lambda + \cdots + \alpha_0 \end{aligned} \tag{15}$$

This enables to impose the poles of the system, poles we choose real and distinct.

Once L_c determined, a simple calculation of $L = L_c P_c^{-1}$ allows to determine the state feedback into the initial base.

It comes for the closed loop initial model the characteristic matrix

$$H(x) \;=\; (A(x) \;\text{-}\; B(x)L) \tag{16}$$

the linearised closed loop system is described as following

$$\dot{x}(t) = H(0)x(t) \tag{17}$$

with

$$H(0) = A(0) - B(0)L \tag{18}$$

A suitable choice of the gain vector L enables to make the poles, of this linear closed loop system, real and distinct.

In practice, a first determination of the attractor can be achieved directly on the initial representation. Another one obtained by the use of the change of basis, which diagonalizes the linearized system at the origin, can lead to different and, very often, better results. With this change of base, the representation of the initial nonlinear system is generally diagonal dominant in the neighborhoods of the origin which enables, with a convenient definition of the comparison system, a better estimation of the attractor.

Let now P be the change of variables which diagonalizes the linearized closed loop model characterized by $H(0)$.

It comes, the corresponding diagonal characteristic matrix $H_d(0)$ such that

$$H_d(0) = P^{-1}H(0)P \tag{19}$$

By using the new state vector x_d , $x_d = \left[x_{d1}, x_{d2} \ldots, x_{dn}\right]^T$, such that

$$x(t) = Px_d(t) \tag{20}$$

$B_d^{'}$, characterizing the uncertainty in the new base, is defined by

$$B_d^{'} = P^{-1}B' \tag{21}$$

it comes for the initial non linear system

$$\dot{x}_d(t) = H_d(.)x_d(t) + B_d^{'}(.) \tag{22}$$

where $H_d = \left\{a_{d_{ij}}(.)\right\}$ is defined by

$$H_d(.) = P^{-1}H(.)P \tag{23}$$

After applying the change of base allowing to diagonalize the linearized system to the initial one's (1), we propose, in this paper, to study the stability and to determine the attractor of the initial system, controlled by the same state feedback law (3).

2.2 Proposed Attractor Determination

For the vector norm $p(x_d) = \left[|x_{d1}|, |x_{d2}| \ldots, |x_{dn}| \right]^T$ (Appendix A), the overvaluing system of the perturbed system is described by [13].

$$\frac{d}{dt} p(x_d) \le M(.) p(x_d) + N(.) \tag{24}$$

where $M(H_d(.)) = \{m_{i,j}(.)\}$ is obtained by replacing the off-diagonal elements of $H_d(x)$ by their absolute values such as

$$\begin{cases} m_{i,i}(.) = a_{d_{i,i}}(.) & \forall i = 1, 2, \ldots n \\ m_{i,j}(.) = \left| a_{d_{i,j}}(.) \right| & \forall i \ne j \end{cases} \tag{25}$$

and $N(.)$ defined by

$$N(.) = \left| B_d^{'}(.) \right| \tag{26}$$

With $M = \max M(.)$ and $N = \max N(.)$, it comes the linear comparison system

$$\dot{z} = Mz + N \tag{27}$$

such that

$$z(t_0) \ge p(x_d(t_0)) \quad \text{implies} \quad z(t) \ge p(x_d(t)), \forall t \succ t_0$$

If *M* is the opposite of an M-matrix, we can have an estimation by overvaluation of the attractor defined by

$$p(x_d(t)) \le -M^{-1} N \tag{28}$$

or

$$p(P^{-1} x(t)) \le -M^{-1} N \tag{29}$$

Then, we have

$$\lim_{t \to +\infty} z(t) = -M^{-1} N \tag{30}$$

and

$$\lim_{t \to +\infty} p(x_d(t)) \le -M^{-1} N \tag{31}$$

It comes the attractor D_1 of system (22) defined by

$$D_1 = \left\{ x_d \in R^n ; p(x_d) \le -M^{-1} N \right\} \tag{32}$$

In the domain D_1, according to the limitations that appear on the state variables, it is possible to the choose a new nonlinear model which enables to determine a better estimation of the attractor as it appears in the application of Section 3

3 Attractor Characterization of a Third Order Nonlinear Complex System

Let us consider the third order system (S) described by

$$(\text{S}): \dot{x}(t) = A(x,t)x(t) + B(x,t)u(t) + B'(.) \tag{33}$$

with

$$A(x(t)) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{34}$$

$$a_{11} = 7$$
$$a_{12} = -2.9 - 0.1e^{-x_2^2}$$
$$a_{13} = -5$$
$$a_{21} = 0.1\sin x_1 + 6\cos x_3$$
$$a_{22} = 1.05 - 2.05\cos x_3$$
$$a_{23} = 5 - 3\cos x_3$$
$$a_{31} = -12 - 0.1\sin x_1$$
$$a_{32} = 0$$
$$a_{33} = -2 + 0.02\sin x_1$$

and

$$B(x(t)) = \begin{bmatrix} -2 \\ -\cos x_3 \\ 2 \end{bmatrix} \tag{35}$$

$$B^{'}(x) = \begin{bmatrix} -0.2\,\text{sat}\, x_1 \\ b_2'(.) \\ 0.1e^{-x_2^2} \end{bmatrix} \tag{36}$$

such that

$$\begin{gathered} \text{sat}\, x_i = x_i, \text{if } |x_i| < 1, \text{ else, sat}\, x_i = \text{sign } x_i, \\ \text{and, } |b_2'(.)| \le 0.15 \end{gathered} \tag{37}$$

By linearization of the system without uncertainties, around the operating point $x_0 = 0$, we obtain the linear model characterized by the following $A(0)$ and $B(0)$

$$A(0) = \begin{bmatrix} 7 & -3 & -5 \\ 6 & -1 & 2 \\ -12 & 0 & -2 \end{bmatrix} \tag{38}$$

and

$$B(0) = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \tag{39}$$

Then, by putting the linearized system in controllable canonical form, it comes

$$\dot{x}_c = A_c x_c + B_c u \tag{40}$$

The characteristic polynomial of the linearized system can be written as

$$\det(\lambda I - A(0)) = \lambda^3 - 4\lambda^2 - 61\lambda - 110 \tag{41}$$

and we have

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 110 & 61 & 4 \end{bmatrix}; \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{42}$$

In order to impose a choosen dynamic to the process, the state feedback gain *L*, of system (17) with (18), (39) and (40), is chosen such that the poles of the closed loop characteristic $\mathrm{P}_{A(0)-B(0)L}(\lambda)$ are (-3), (-4) and (-5), i.e the characteristic polynomial:

$$\begin{aligned} \mathrm{P}_{A(0)-B(0)L}(\lambda) &= (\lambda+3)(\lambda+4)(\lambda+5) \\ &= \lambda^3 + 12\lambda^2 + 47\lambda + 60 \end{aligned} \tag{43}$$

corresponding to the following characteristic matrix H_c

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 60 & 47 & 12 \end{bmatrix} \tag{44}$$

The state feedback gain have to satisfy the following conditions

$$u = -\begin{bmatrix} 170 & 108 & 16 \end{bmatrix} x_c = \begin{bmatrix} l_{c_1} & l_{c_2} & l_{c_3} \end{bmatrix} x_c \tag{45}$$

Given that we have $L = L_c P_c^{-1}$, it comes the control vector gain

$$L = \begin{bmatrix} -6 & 2 & 3 \end{bmatrix} \tag{46}$$

and the matrix of the closed loop system without uncertainties linearized at the origin $H(0)$

$$H(0) = \begin{bmatrix} -5 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & -4 & -8 \end{bmatrix} \tag{47}$$

which becomes diagonal for the change of base P defined by

$$P = \begin{bmatrix} 0.125 & 0 & -0.625 \\ 1.25 & 0.75 & 0 \\ -1 & -0.75 & 0 \end{bmatrix} \tag{48}$$

In this case, the initial system defined by (33) with (34) and (35), controlled by the control law (3) with (47), can be described by $H(x) = (A(x) - B(x)L)$ such that

$$H(x(t)) = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \tag{49}$$

with

$$h_{11} = -5$$

$$h_{12} = 1.1 - 0.1e^{-x_2^2}$$

$$h_{13} = 1$$

$$h_{21} = 0.1\sin x_1$$

$$h_{22} = 1.05 - 0.05\cos x_3$$

$$h_{23} = 5$$

$$h_{31} = -0.1\sin x_1$$

$$h_{32} = -4$$

$$h_{33} = 0.02\sin x_1 - 8$$

Let us try to determine directly an attractor estimation D_1 of the initial model.

If the comparison system of the process is in the form (27), according to (49), the minimal overvaluing matrix relatively to the regular vector norm $p(x) = \left[|x_1|,|x_2|,|x_3|\right]^T$ is

$$M(H(x(t))) = \begin{bmatrix} h_{11} & |h_{12}| & |h_{13}| \\ |h_{21}| & h_{22} & |h_{23}| \\ |h_{31}| & |h_{32}| & h_{33} \end{bmatrix} \tag{50}$$

and $N(B^{'})$ is

$$N(B^{'}) = \begin{bmatrix} 0.2 \\ 0.15 \\ 0.1 \end{bmatrix} \tag{51}$$

In this case, the comparison system can be described by

$$\dot{z} = \begin{bmatrix} -5 & 1.1 & 1 \\ 12.1 & 1.1 & 5 \\ 0.1 & -4 & -7.98 \end{bmatrix} z + \begin{bmatrix} 0.2 \\ 0.15 \\ 0.1 \end{bmatrix} \tag{52}$$

For this comparison system, the matrix *M* is not the opposite of an M-matrix because of one of the diagonal elements is positive. Then we cannot conclude concerning the determination of an attractor.

By the use of change of variables *P*, $H_d(x)$ becomes such that

$$H_d(x(t)) = \begin{bmatrix} h_{d11} & h_{d12} & h_{d13} \\ h_{d21} & h_{d22} & h_{d23} \\ h_{d31} & h_{d32} & h_{d33} \end{bmatrix} \tag{53}$$

with

$$h_{d11} = -0.25\cos x_3 - 0.08\sin x_1 - 2.75$$
$$h_{d12} = -0.15\cos x_3 - 0.06\sin x_1 + 0.15$$
$$h_{d13} = 0$$
$$h_{d21} = 0.33\cos x_3 + 0.15\sin x_1 - 0.333$$
$$h_{d22} = 0.2\cos x_3 + 0.1\sin x_1 - 4.2$$
$$h_{d23} = -0.0833\sin x_1$$
$$h_{d31} = 0.2e^{-x_2^2} - 0.05\cos x_3 - 0.016\sin x_1 - 0.15$$
$$h_{d32} = 0.12e^{-x_2^2} - 0.03\cos x_3 - 0.012\sin x_1 - 0.09$$
$$h_{d33} = -5$$

The comparison system of the process, corresponding to the vector norm $p(x_d) = \left[|x_{d1}|, |x_{d2}|, |x_{d3}|\right]^T$, is in the form (27), with

$$N = \max | P^{-1} B^{'}(.) | \le \begin{bmatrix} 1 \\ 1.4667 \\ 0.733 \end{bmatrix} \tag{55}$$

According to (49), the minimal overvaluing matrice relatively to the regular vector norm is the following

$$M = \begin{bmatrix} -2.42 & 0.36 & 0 \\ 0.813 & -3.9 & 0.0833 \\ 0.216 & 0.132 & -5 \end{bmatrix}$$
$$\text{and } N = \begin{bmatrix} 1 \\ 1.4667 \\ 0.733 \end{bmatrix} \tag{56}$$

It is trivial that the following conditions

$$\begin{cases} -2.42 < 0 \\ (-2.42 \times -3.9) - (0.813 \times 0.36) > 0 \\ \det(M) < 0 \end{cases} \tag{57}$$

are satisfied, M is then the opposite of an M-matrix (Appendix B),

and we have

$$\lim_{t \to +\infty} z(t) = -M^{-1} N \tag{58}$$

and

$$\lim_{t \to +\infty} p(x_d(t)) \le -M^{-1} N \tag{59}$$

It comes an estimation, by overvaluation, of the attractor defined by $p(x_d(t)) \le -M^{-1} N$, or

$$p(x_d(t)) \le \begin{bmatrix} 0.4848 \\ 0.4810 \\ 0.1802 \end{bmatrix} \tag{60}$$

The attractor D_1 is finally defined by

$$\begin{cases} |4x_2 + 4x_3| \le 0.4848 \\ |-5.333x_2 - 6.6667x_3| \le 0.4810 \\ |-1.6x_1 + 0.8x_2 + 0.8x_3| \le 0.1802 \end{cases} \tag{61}$$

In D_1 we have $|x_1| \le 0.1732$, $|x_2| \le 0.9643$ and $|x_3| \le 0.8431$

A new description of the system (S) can be defined, in D_1

As $|x_1| \le 0.1732$ it comes, $\text{sat}\, x_1 = x_1$, then this value can be introduced in the definition of $H(x(t))$

Hence the description

$$H(x(t)) = \begin{bmatrix} -5.2 & 1.1 - 0.1e^{-x_2^2} & 1 \\ 0.1\sin x_1 & 1.05 - 0.05\cos x_3 & 5 \\ -0.1\sin x_1 & -4 & 0.02\sin x_1 - 8 \end{bmatrix} \tag{62}$$

and

$$B^{'}(x) = \begin{bmatrix} 0 \\ b_2'(.) \\ 0.1e^{-x_2^2} \end{bmatrix} \tag{63}$$

By the use of change of variables P, $H_d(x)$ becomes such that

$$H_d(x(t)) = \begin{bmatrix} h_{d11}^{'} & h_{d12}^{'} & h_{d13}^{'} \\ h_{d21}^{'} & h_{d22}^{'} & h_{d23}^{'} \\ h_{d31}^{'} & h_{d32}^{'} & h_{d33}^{'} \end{bmatrix} \tag{64}$$

with

$$h_{d11}^{'} = -0.25\cos x_3 - 0.08\sin x_1 - 2.75$$
$$h_{d12}^{'} = -0.15\cos x_3 - 0.06\sin x_1 + 0.15$$
$$h_{d13}^{'} = 0$$
$$h_{d21}^{'} = 0.33\cos x_3 + 0.15\sin x_1 - 0.333$$
$$h_{d22}^{'} = 0.2\cos x_3 + 0.1\sin x_1 - 4.2$$
$$h_{d23}^{'} = -0.0833\sin x_1$$
$$h_{d31}^{'} = 0.2e^{-x_2^2} - 0.05\cos x_3 - 0.016\sin x_1 - 0.11$$
$$h_{d32}^{'} = 0.12e^{-x_2^2} - 0.03\cos x_3 - 0.012\sin x_1 - 0.09$$
$$h_{d33}^{'} = -5.2$$

the comparison system corresponding to the vector

$p(x_d) = \left[|x_{d1}|, |x_{d2}|, |x_{d3}|\right]^T$, is in the form (26), with

$$N = \max | P^{-1}B^{'}(.) | \leq \begin{bmatrix} 1 \\ 1.4667 \\ 0.2 \end{bmatrix} \quad (65)$$

Then, in D_1, the comparison system of the process is on the form (27). According to (64), the minimal overvaluing matrices relatively to the regular vector norm are the followings

$$M = \begin{bmatrix} -2.9025 & 0.0605 & 0 \\ 0.1393 & -3.9828 & 0.0144 \\ 0.0897 & 0.0783 & -5.2 \end{bmatrix}$$
$$\text{and } N = \begin{bmatrix} 1 \\ 1.4667 \\ 0.2 \end{bmatrix} \quad (66)$$

As the following conditions

$$\begin{cases} -2.9025 < 0 \\ (-2.9025 \times -3.9828) - (0.1393 \times 0.0605) > 0 \\ \det(M) < 0 \end{cases} \quad (67)$$

are satisfied, M is, then, the opposite of an M-matrix.

It comes an estimation, by overvaluation, of the attractor defined by $p(x_d(t)) \leq -M^{-1}N$

or

$$p(x_d(t)) \leq \begin{bmatrix} 0.3525 \\ 0.3808 \\ 0.0503 \end{bmatrix} \quad (68)$$

The attractor D_2 is finally defined by

$$\begin{cases} |4x_2 + 4x_3| \leq 0.3525 \\ |-5.333x_2 - 6.6667x_3| \leq 0.3808 \\ |-1.6x_1 + 0.8x_2 + 0.8x_3| \leq 0.0503 \end{cases} \quad (69)$$

The obtained attractors D_1 and D_2 are given in the Figure 1, for which a trajectory in the state space is simulated for $b_2'(.) = 0.15\sin t$.

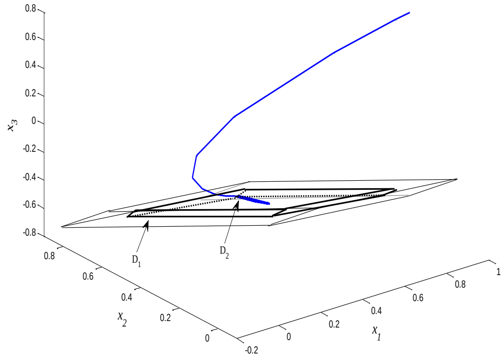

Figure 1
Evolution of the state vector towards the attractors D_1 and D_2 (in bold)

Conclusion

An efficient technique for determination of attractors characterizing the precision of a control law is defined in this paper using the concept of vector norm, associated to the definition of comparison systems obtained by the use of the Borne and Gentina stability approach. The proposed approach for determination of the control law by state or output feedback in presence of uncertainties is based on a local linearization and control of the system. Process control through poles placement of the linearized system is used in the feedback control. This method enables to test the accuracy of a controlled system by providing an estimation by overvaluation of the error. The proposed method is applied with success for a third order nonlinear complex system to illustrate the efficiency of the proposed approach.

Appendices

Appendix A. Vector Norms Definition

Definition1: Let $\mathrm{E} = \mathrm{R}^n$ and $\mathrm{E}_1, \mathrm{E}_2 \ldots \mathrm{E}_k$ be subspaces of the space E, $\mathrm{E} = \mathrm{E}_1 \cup \mathrm{E}_2 \ldots \cup \mathrm{E}_k$

Let x be an n vector defined on E and $x_i = P_i x$ the projection of x on E_i , where P_i is a projection operator from E into E_i, p_i a scalar norm (i=1,2,…, k) defined on the subspace E_i and p denotes a vector norm of dimension k and with its component

$$p_i(x) = p_i(x_i) , \quad p(x): R^n \rightarrow R_+^k$$

Let *y* be another vector in space E, with $y_i = P_i y$, we have the following properties

$$\begin{cases} p_i(x_i) \geq 0, \forall x_i \in \mathrm{E_i} \forall i = 1,2,\ldots,\mathrm{k} \\ p_i(x_i) = 0 \leftrightarrow x_i = 0, \forall i = 1,2,\ldots,\mathrm{k} \\ p_i(x_i + y_i) \leq p_i(x_i) + p_i(y_i), \forall x_i, y_i \in \mathrm{E_i} \forall i = 1,2,\ldots,\mathrm{k} \\ p_i(\lambda x_i) = |\lambda|\, p_i(x_i), \forall x_i \forall i = 1,2,\ldots,\mathrm{k}, \forall \lambda \cup R \end{cases}$$

If k-1 of the subspaces E_i are insufficient to define the whole space E, the vector norm is surjective.

If in addition the subspaces E_i are in disjoint pairs, $\mathrm{E}_i \cap \mathrm{E}_j = \varnothing$, $\forall i \neq j = 1,2,\ldots,\mathrm{k}$, the vector norm p

is said to be regular.

Appendix B. Overvaluing and comparison systems

Let the differential equation $\dot{x} = A(x,t)x$. The overvaluing system is defined by the use of the vector norm $p(x)$ of the state vector *x* and the use of the right-band derivation $D^+ p_i(x_i)$ proposed by [28, 29] $D^+ p_i(x_i)$ is taken along the motion of *x* in the subspace E_i and $D^+ p(x)$ along the motion of *x* in E.

Definition 2: The matrix $M(x,t)$ defines an overvaluing system of S with respect to the vector norm *p* if and only if the following inequality is verified for each corresponding component: $D^+ p(x) \leq M(x,t)\, p(x)$

If for the same system we can define a constant overvaluing matrix *M*, we have $M \geq M(x,t)$ and we have $z(t) \geq p(x(t))$ for $t \geq t_0$ as soon as this property is satisfied at the origin t_0

When an overvaluing matrix $M(x,t)$ of a matrix $A(x,t)$ is defined with respect to a regular vector norm *p* we have the following properties:

- The off- diagonal elements of matrix $M(x,t)$ are non negative.
- If we denote by $\mathrm{Re}(\lambda_M)$ the real part of the eigenvalue of the maximum real part of $M(x,t)$ the following inequality is verified

  $\mathrm{Re}(\lambda_A) \leq \mathrm{Re}(\lambda_M) = \lambda_M \quad \forall t, x \in \tau \times \mathbb{R}^n$,

  whatever the eigenvalue λ_A of matrix $A(x,t)$

- When all the real parts of the eigenvalues of M (x, t) are negative this matrix is the opposite of an M-matrix and it admits an inverse whose elements are all non positive.
- When due to perturbations and/or uncertainties it is not possible to define an homogeneous overvaluing system we can define a non homogeneous overvaluing system of the form $D^+ p(x) \leq M(x,t)\, p(x) + N(x,t)$, where all the elements of vector norm nonnegative and when M and N are constant, we can define the comparison system $\dot{z} = Mz + N$

Remark 1. With $M(.) = \{m_{ij}(.)\}$ the verification of the Kotelyanski lemma by the matrix $M(.)$ prove that $M(.)$ is the opposite of an M-matrix

$$m_{1,1} \prec 0, \begin{vmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{vmatrix} \succ 0, \ldots, (-1)^k \begin{vmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,k} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ m_{k,1} & m_{k,2} & \cdots & m_{k,k} \end{vmatrix} \succ 0$$

Remark 2. A less conservative approach consists to use a vector norm of size k=n, for example $p(x) = \left[|x_1|, |x_2|, \ldots, |x_n|\right]^T$

Remark 3. If $M(.)$ is an overvaluing matrix of a matrix $A(.)$, $M(.) + M^*$ where the elements of M^* are all non negative is also an overvaluing matrix of $A(.)$. This property can be used to simplify the determination of an overvaluing matrix of $A(.)$ when some elements of $A(.)$ are ill defined or subject to uncertainties.

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