# Variations of non-additive measures

**Endre Pap**

Department of Mathematics and Informatics, University of Novi Sad
Trg D. Obradovica 4, 21 000 Novi Sad, Serbia and Montenegro
e-mail: pape@eunet.yu

*Abstract: General non-additive measures are investigated with the help of some related monotone measures (some types of variations and submeasures), which have some important additional properties.*

*Keywords: Set function, variation, submeasure.*

## 1 Introduction

Non-additive set functions, as for example outer measures, semi-variations of vector measures, appeared naturally earlier in the classical measure theory concerning countable additive set functions or more general finite additive set functions. The pioneer in the theory of non-additive set functions was G.Choquet [2] from 1953 with his theory of capacities. This theory had influences on many parts of mathematics and different areas of sciences and technique.

Non-additive set functions are extensively used in the decision theory, mathematical economy, social choice problems, with early traces by Aumann and Shaplay in their monograph [1]. Recently, many authors have investigated different kinds of non-additive set functions, as subadditive and superadditive set functions, submeasures, k-triangular set functions, t-conorm and pseudo-addition decomposable measures, null-additive set functions , and many other types of set functions. Although in many results the monotonicity of the observed set functions was supposed, there are some results concerning also some classes of set functions which include also non-monotone set functions (for example superadditive set functions, k-triangular set functions).

On the other hand, fuzzy measures as monotone and continuous set functions were investigated by Sugeno [17] in 1974 with the purpose to evaluate non-additive quantity in systems engineering.This notion of fuzziness is different from the one given by Zadeh.Namely, instead of taking membership grades of a set, we take (in the fuzzy measure approach) the measure that a given unlocated element belongs to a set. There are many different type of fuzzy measures which are used . For example belief, possibility, decomposable measures. Specially in different branches of mathematics there are many types of non-additive set functions. They appeared in the potential theory, harmonic analysis, fractal geometry, functional analysis, theory of nonlinear differential equations, theory of difference equations and optimizations. There are many different fields in which the interest on non-additive set functions is growing up. In the theory of the artificial intelligence, belief functions have been applied to model uncertainty. Belief functions, corresponding plausibility measures and other kinds of non-additive set functions are used in statistics. Non-additive expected utility theory has been applied for example in multi-stage decision and economics. Many aggregation operators are based on integrals related to non-additive measures [9, 10, 12]. We can compare additive set functions (which are base for the classical measure theory) and non-additive set functions in the following simple way. For a fixed set $A \in \Sigma$ the classical measure $\mu : \Sigma \to [0, +\infty]$ gives that for every set $B$ from $\Sigma$ such that $A \cap B = \emptyset$ we have that $\mu(A \cup B) - \mu(B)$ is always equal to a constant $\mu(A)$, i.e., it is independent of $B$. In contrast, for non-additive set function $m$ the difference $m(A \cup B) - m(B)$ depends on $B$ and can be interpreted as the effect of $A$ joining $B$.

In this paper we will correspond to every set function ([1, 2, 13, 14, 15, 17, 18]) special positive set functions with some additional properties. Motivated by the notion of the variation of the classical measure ([15, 16]) we introduce axiomatically the notion of the variation of the general set function and prove that it always exists, but in general case it is not unique. One of them so called disjoint variation is based on the partition of the set and the other so called chain variation is based on the chains of sets, see [1, 4, 14]. In this paper we will prove that these variations have some additional properties with respect to the starting non-additive set function. Among others that disjoint variation is superadditive on any family of disjoint sets, see [14].

## 2 Variations

We start with some results from the classical measure theory. Let $\Sigma$ be a $\sigma$-algebra of subsets of the given set $X$. A set function $\mu : \Sigma \to \mathbb{R}$ is additive (signed finitely additive measure) if we have

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all $A, B \in \Sigma$ with $A \cap B = \varnothing$. A set function $\mu : \Sigma \to \mathbb{R}$ is $\sigma-$additive (signed measure) if we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

for all pairwise disjoint sequences $\{E_i\}$ from $\Sigma$, i.e., $E_n \cap E_m = \varnothing$ for $n \neq m$.

For an arbitrary but fixed subset $A$ of $X$ and an additive set function $\mu$ its variation $\overline{\mu}$ is defined by

$$\overline{\mu}(A) = \sup_I \sum_{i \in I} \mid \mu(D_i) \mid,$$

where the supremum is taken over all finite families $\{D_i\}_{i\in I}$ of pairwise disjoint sets of $\Sigma$ such that $\cup_{i\in I} D_i = A$. It is well-known that if $\mu : \Sigma \to \mathbb{R}$ is finitely (or countable) additive the $\overline{\mu}$ is finitely (countable) additive. If $\mu$ is signed additive set function then $\mu$ is countable additive if and only if $\overline{\mu}$ is countable additive.

We consider now general set functions $m$, $m : \mathcal{D} \to [-\infty, +\infty]$, with $m(\varnothing) = 0$ (extended real-valued set function), see [14], where $\mathcal{D}$ denote a family of subsets of a set $X$ with $\varnothing \in \mathcal{D}$. $m$ is (finite) real-valued set function if $-\infty < m(A) < +\infty$ for all $A \in \mathcal{D}$, and $m$ is monotone if $A \subset B$ implies $m(A) \le m(B)$ for every $A, B \in \mathcal{D}$. $m$ is non-negative if it is finite and $m(A) \ge 0$ for all $A \in \mathcal{D}$, and $m : \mathcal{D} \to [0, +\infty]$ is positive.

We introduce for an arbitrary set function axiomatically a generalization of the variation.

**Definition 1** *Let $m$ be a set function defined on $\mathcal{D}$ with values in $\mathbb{R}$ (or $[0, +\infty]$), with $m(\varnothing) = 0$. Then variation of $m$ is a set function $\eta : \mathcal{D} \to [0, +\infty]$ with the following properties:*

*(i) For every $A \subset X$ we have*

$$0 \le \eta(A) \le +\infty;$$

*(ii)* $\eta(\varnothing) = 0$;

*(iii)* $| \, m(A) \, | \le \eta(A) \quad (A \in \mathcal{D})$;

*(iv) $\eta$ is monotone, i.e., if $B \subset A$, then $\eta(B) \le \eta(A)$;*

*(v) $\eta(A) = 0$ if and only if $m(B) = 0$ for every subset $B$ of $A$ from $\mathcal{D}$.*

We easily obtain: For every $A \subset X$ we have

$$\eta(A) \ge \sup\{| \, m(B) \, | : \; B \subset A, \; B \in \mathcal{D}\}.$$

Namely, if $B$ is a arbitrary subset of $A$ which belongs to $\mathcal{D}$ we have by the properties (iv) and (iii)

$$\eta(A) \ge \eta(B) \ge | \, m(B) \, | \, .$$

# 3 The existence of variations

**Theorem 1** *For every set function $m$ defined on $\mathcal{D}$ and with values in $\mathbb{R}$ (or $[0, +\infty]$), with $m(\varnothing) = 0$, always exists its variation, which in general case is not uniquely determined.*

We introduce two special set functions related to a given set function $m$ which are base for the proof of the preceding theorem.

**Definition 2** *For an arbitrary but fixed subset $A$ of $X$ and a set function $m$ we define the disjoint variation $\overline{m}$ by*

$$\overline{m}(A) = \sup_I \sum_{i \in I} | \, m(D_i) \, |, \tag{1}$$

*where the supremum is taken over all finite families $\{D_i\}_{i\in I}$ of pairwise disjoint sets of $\mathcal{D}$ such that $D_i \subset A \ (i \in I)$.*

**Definition 3** *For an arbitrary but fixed $A \in \mathcal{D}$ and a set function $m$ we define the chain variation $\mid m \mid$ by*

$$\mid m \mid (A) = \sup\{\sum_{i=1}^{n} \mid m(A_i) - m(A_{i-1}) \mid :$$

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = A, A_i \in \mathcal{D}, i = 1, \ldots, n\}. \tag{2}$$

We remark that the supremum in the previous definition is taken over all finite chains between $\varnothing$ and $A$.

**Proof of Theorem 1.** Let $m : \mathcal{D} \to \mathbb{R}$. We prove that $\overline{m}$ and $\mid m \mid$ are variations.

First we consider $\overline{m}$:

(i) Follows by the definition of $\overline{m}$.

(ii) Only family is $I = \{\varnothing\}$, and so we have

$$\overline{m}(\varnothing) = \sup \Sigma \mid m(\varnothing) \mid = 0.$$

(iii) Since one of the families $I$ of disjoint sets of $\mathcal{D}$ contained in $A$ is the family $I = \{A\}$ we obtain by the definition of $\overline{m}$ the desired inequality.

(iv) Since for any family $\{D_i\}_{i\in I}$ of disjoint sets of $\mathcal{D}$ with property $D_i \subset B \ (i \in I)$, by $B \subset A$, we have $D_i \subset A$, too, we obtain

$$\sum_{i\in I} \mid m(D_i) \mid \leq \overline{m}(A),$$

and so $\overline{m}(B) \leq \overline{m}(A)$.

(v) Let us suppose that $\overline{m}(A) = 0$ for an arbitrary set $A \subset X$. Then by (v) we obtain

$$\overline{m}(B) = 0 \quad \text{for each} \quad B \subset A, \ \ B \in \mathcal{D}.$$

Suppose now that $\overline{m}(B) = 0$ for each $B \subset A, \ B \in \mathcal{D}$. Then for each finite family $\{D_i\}_{i\in I}$ of disjoint sets such that $D_i \subset A \ (i \in I)$ we obtain

$$\sum_{i\in I} \mid m(D_i) \mid = 0.$$

This implies $\overline{m}(A) = 0$.

We consider now $\mid m \mid$:

(i) Follows by (2).

(ii) Every chain connecting $\varnothing$ with $\varnothing$ consists only of empty set, therefore

$$\begin{aligned} \mid m \mid (\varnothing) &= \sup\{\sum \mid m(A_i) - m(A_{i-1}) \mid : \\ &\quad \varnothing = A_0 \subset A_n = \varnothing\} = 0. \end{aligned}$$

(iii) One of the chain which connects $\varnothing$ and the set $A$ consists only of $\varnothing$ and $A$, i.e., $\varnothing = A_0 \subset A_n = A$, therefore

$$\begin{aligned} \mid m(A) \mid &= \sup\{\mid m(A) - m(\varnothing) \mid : \\ \varnothing = A_0 \subset A_n = A\} &\leq \mid m \mid (A). \end{aligned}$$

(iv) Some of chains from $\varnothing$ to $A$ contain chains from $\varnothing$ to $B$, (exhaust all of them), therefore

$$\begin{aligned} \mid m \mid (B) = \sup\{\sum_{i=1}^{n} \mid m(A_i) - m(A_{i-1}) \mid : \\ \varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = B, A_i \in \mathcal{D},\ i = 1, ..., n\} \\ \leq \sup\{\sum_{i=1}^{s} \mid m(A_i) - m(A_{i-1}) \mid : \\ \varnothing \subset A_0 \subset A_1 \subset \cdots \subset A_s = A, A_i \in \mathcal{D},\ \ i = 1, ..., s\} \\ = \mid m \mid (A). \end{aligned}$$

(v) Suppose $\mid m \mid (A) = 0$. Then for every set $B \subset A,\ B \in \mathcal{D}$, we have by (v) $m(B) = 0$.

Conversely, if $m(B) = 0$ for every set $B \subset A,\ B \in \mathcal{D}$, then for any chain $\varnothing = A_0 \subset A_1 \subset ... \subset A_n = A$ we have $m(A_i) = 0,\ i = 1, 2, ..., n$, and so

$$\sum_{i=1}^{n} \mid m(A_i) - m(A_{i-1}) \mid = 0.$$

Hence $\mid m \mid (A) = 0$.

Finally we shall show that generally $\overline{m} \neq \mid m \mid$ . Let $X = \{a, b\}$. Define $m : \mathcal{P}(X) \to \mathbb{R}$ by $m(\{a\}) = 2, m(\{b\}) = 3, m(\{a, b\}) = 4$. Then $\overline{m}(\{a, b\}) = 5 \neq 4 = \mid m \mid (\{a, b\})$. □

**Remark 1** (i) If $\mathcal{D}$ is an algebra, then we can take for $A \in \mathcal{D}$ in (1) the supremum for all finite families $\{D_i\}_{i \in I}$ of disjoint sets such that $\bigcup_{i \in I} D_i = A$.
(ii) We note that the variation $\overline{m}$ given by (1) is defined on $\mathcal{P}(X)$.

Let $\mathcal{D}$ be a ring. A set function $m : \mathcal{D} \to \mathbb{R}$ is superadditive if for every $A, B \in \mathcal{D}$ with $A \cap B = \varnothing$ we have $m(A \cup B) \geq m(A) + m(B)$, and it is subadditive if for every $A, B \in \mathcal{D}$ with $A \cap B = \varnothing$ we have $m(A \cup B) \leq m(A) + m(B)$.

**Theorem 2** *Let $m$ be a set function defined on $\Sigma$ with values in $\mathbb{R}$ (or $[0, +\infty]$), with $m(\varnothing) = 0$. Then the set function $\overline{m}$ given by (1) is superadditive, i.e.,*

$$\sum_{i \in I} \overline{m}(E_i) \leq \overline{m}\left(\bigcup_{i \in I} E_i\right)$$

*for each family $\{E_i\}_{i \in I}$ of disjoint sets of $X$.*

**Proof.** Suppose that $m \neq 0$. We take two arbitrary but fixed disjoint subset $E_1$ and $E_2$ of $X$ such that $\overline{m}$ is different from zero on at least one of them. We take an arbitrary but fixed real number $r$ such that

$$\overline{m}(E_1) + \overline{m}(E_2) > r. \tag{3}$$

Therefore there exist two real numbers $r_1$ and $r_2$ such that $r = r_1 + r_2$ and

$$\overline{m}(E_1) > r_1 \quad \text{and} \quad \overline{m}(E_2) > r_2.$$

Then there exists a finite family $\{D_i^1\}_{1 \leq i \leq n}$ of disjoint sets from $\Sigma$ with $D_i^1 \subset E_1$ $(i = 1, 2, \ldots, n)$ such that

$$\sum_{i=1}^{n} \mid m(D_i^1) \mid > \ r_1, \tag{4}$$

and a finite family $\{D_i^2\}_{1 \leq j \leq k}$ of disjoint sets from $\Sigma$, with $D_j^2 \subset E_2$ $(j = 1, 2, \ldots, k)$, such that

$$\sum_{j=1}^{k} \mid m(D_j^2) \mid > \ r_2. \tag{5}$$

The family of sets

$$\{D_1^1, D_2^1, \ldots, D_n^1, D_1^2, D_2^2, \ldots, D_k^2\}$$

consists of disjoint sets from $\Sigma$ which are contained in $E_1 \cup E_2$ and by (1) we obtain

$$\overline{m}(E_1 \cup E_2) \geq \sum_{i=1}^{n} \mid m(D_i^1) \mid + \sum_{j=1}^{k} \mid m(D_j^2) \mid .$$

Therefore by the inequalities (4) and (5)

$$\overline{m}(E_1 \cup E_2) > r_1 + r_2 = r. \tag{6}$$

Since the number $r$ was arbitrary such that the inequality (3) is satisfied, we conclude that

$$\overline{m}(E_1) + \overline{m}(E_2) \leq \overline{m}(E_1 \cup E_2)$$

since the opposite inequality is impossible by (6).

The preceding inequality holds by induction for every finite family $\{E_i\}_{i \in J}$ of disjoint subsets of $X$, i.e.,

$$\sum_{i \in J} \overline{m}(E_i) \leq \overline{m}\left(\bigcup_{i \in J} E_i\right).$$

For an arbitrary family $\{E_i\}_{i \in I}$ of disjoint subset of $X$ we obtain by the preceding step that for each finite subset $J$ of $I$

$$\overline{m}\left(\bigcup_{i \in I} E_i\right) \geq \overline{m}\left(\bigcup_{i \in J} E_i\right) \geq \sum_{i \in J} \overline{m}(E_i).$$

Therefore

$$\overline{m}\left(\bigcup_{i\in I} E_i\right) \geq \sum_{i\in I} \overline{m}(E_i).$$

□

**Open problem**: *Find all variations of a given arbitrary set function $m$.*

We shall give a partial answer on this problem, when we require some additional properties of the variation.

**Theorem 3** *Let $m$ be a set function defined on $\Sigma$ with values in $\mathbb{R}$ (or $[0,+\infty]$), with $m(\varnothing) = 0$. Then $\overline{m}$ given by (1) is the smallest variation of $m$ (defined on $\mathcal{P}(X)$) which is superadditive.*

**Proof.** By Theorem 2 variation $\overline{m}$ is superadditive. Then theorem follows by the properties of any superadditive variation $\eta$ of $m$ taking an arbitrary finite family $\{D_i\}_{i\in I}$ of disjoint sets from $\Sigma$ contained in $A$, i.e.,

$$\eta(A) \geq \eta\left(\bigcup_{i\in I} D_i\right) \geq \sum_{i\in I} \eta(D_i) \geq \sum_{i\in I} \mid m(D_i) \mid .$$

Hence $\eta(A) \geq \overline{m}(A)$. □

**Open problem**: *Find all variations of a given arbitrary set function $m$.*

We shall give a partial answer on this problem, when we require some additional properties of the variation.

**Theorem 4** *Let $m$ be a set function defined on $\Sigma$ with values in $\mathbb{R}$ (or $[0,+\infty]$), with $m(\varnothing) = 0$. Then $\overline{m}$ given by (1) is the smallest variation of $m$ (defined on $\mathcal{P}(X)$) which is superadditive.*

# 4 Submeasures

For non-negative monotone set function $m$ with an additional topological property we can correspond a submeasure $\xi$ (monotone and subadditive set function) which is closely topologically connected with $m$, see for more details [6, 7, 13, 14].

**Theorem 5** *Let $\mathcal{D}$ be a ring and $m : \mathcal{D} \to [0,\infty)$ be monotone. Then there exists a submeasure $\xi$ on $\mathcal{D}$ such that*

$$m(E_n) \to 0 \Leftrightarrow \xi(E_n) \to 0,$$

*if and only if $m$ satisfies the following condition:*

$$(ac) \quad m(A_n) + m(B_n) \to 0, \quad \textit{then} \quad m(A_n \cup B_n) \to 0.$$

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