10.1515/jbe-2015-0009

Bracing Zonohedra With Special Faces

Gyula Nagy KEM

Institute of Civil Engineering, Szent István University, Budapest, Hungary
nagy.gyula@ybl.szie.hu

Abstract: *The analysis of simpler preliminary design gives useful input for more complicated three-dimensional building frame structure. A zonohedron, as a preliminary structure of design, is a convex polyhedron for which each face possesses central symmetry. We considered zonohedron as a special framework with the special assumption that the polygonal faces can be deformed in such a way that faces remain planar and centrally symmetric, moreover the length of all edges remains unchanged. Introducing some diagonal braces we got a new mechanism. This paper deals with the flexibility of this kind of mechanisms, and investigates the rigidity of the braced framework. The flexibility of the framework can be characterized by some vectors, which represent equivalence classes of the edges. A necessary and sufficient condition for the rigidity of the braced rhombic face zonohedra is posed. A real mechanical construction, based on two simple elements, provides a CAD prototype of these new mechanisms.*

Keywords: *rigidity, bar any joint framework, zonohedra, bracing, optimization*

1. INTRODUCTION

For building frame design, it is useful to consider various frame systems in order to compute the possible motions and the rigidity of the framework. Simple models are useful for involving the interaction of different structural elements of the systems. The analysis of simpler preliminary design gives useful input for more complicated three-dimensional structure. Polyhedra have investigated engineers and scientists for centuries. Several definitions of polyhedra can be found in the literature. They have often been defined as solids. Cauchy, however, considered a polyhedron as a set of rigid faces connected by revolute joints [1]. In this sense every convex polyhedron, is rigid. The first correct proof of this fact was given by Dehn [2], that uses the results of Legendre [3] and Cauchy [1]. A non-convex polyhedra can be, on the contrary, flexible, e.g. Connelly found some non-rigid polyhedra in his paper [4]. On the other hand, the extensive class of nonconvex zonohedra considered by Dolbilin is also rigid [5]. If the polyhedron is defined as bar-and-joints framework and the faces are triangulated with added rods, than almost all framework is rigid [4, 6]. By definition, a zonohedron is a convex polyhedron where every face is a polygon with point symmetry. (Any zonohedron may equivalently be described as the Minkowski sum of a set of line segments.) We consider a zonohedron as a bar-and-joint framework (the edges correspond to bars, while the vertices correspond to joints, contrary to Cauchy and Dolbilin; where the faces are rigid, and they are connected by revolute joints, as mentioned earlier) with the following assumption. The polygonal faces can move such that the faces remain planar and centrally symmetric, (hence the faces there are not rigid in our frameworks). These frameworks, the so-called “zonohedra with special faces” (ZSF in the following) are mechanisms. In this paper we introduce and study bracing of a ZSF. The flexibility of a ZSF will be characterized by vectors, representing equivalence classes of a ZSF. We can make the frameworks rigid using some diagonal braces. We conclude a theorem for the braced zonohedra that provides necessary and sufficient condition for the rigidity of the framework. We also give a real mechanical construction based on simple elements. We note that similar constraints of polyhedron frameworks was considered in [7-10]. The models used to describe the motions of a framework are based on results of one-dimensional and two-dimensional rigidity analyses, respectively. In [11] the Authors use the concept of "zonohedra with articulated faces (ZAF)" as alternative the special assumption used in [7].

2. INFINITESIMAL RIGIDITY OF THE FRAMEWORK

2.1 Rigidity and infinitesimal rigidity of bar-and-joint frameworks

The position of the joints X and Y are denoted by p(x), p(y).
Definition 2.1: The framework is rigid if every continuous motion of its vertices which preserves the lengths of all edges, also preserves the distances between all pairs of vertices.
The concept of rigidity and infinitesimal rigidity are closely related. In space the infinitesimal motion q of a bar-and-joint framework F(p) is a map q: $R^3 \rightarrow R^3$ $q: R^2 \rightarrow R^2$ that can be formulated with the constraint of the Figure 1, if there is a bar between joints X and Y. It is obvious that a rigid body motion satisfies the constraint on Figure 1.

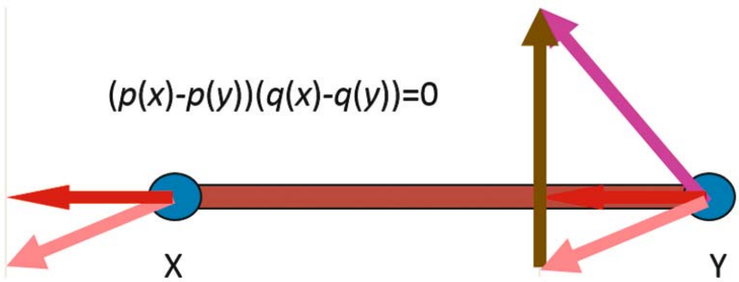

Fig.1. The bar XY between joints X and Y is perpendicular to the difference of the infinitesimal motion vectors in the case of an optimal bar i.e. the bar lengths and bar joint incidences must be preserved. The p(X) and q(X) denote the position and the infinitesimal displacement of X respectively

We can determine the infinitesimal rigidity of a bar-and-joint framework by Maxwell as a rank condition of the rigidity matrix. In case of a cubic framework (Fig.2) we consider the equation system, which describes the possible infinitesimal motion of the joint of the cubic framework. Clearly, the number of the equations is less than the number of variables. One of its reasons is that there are too few bars in the framework, the other reason is some rigid body like motion. Infinitesimal rigidity was most exactly investigated and described in [12,13]. In some special cases, like ours will be, we can work with graph or matroid theoretical models. Their advantage is, that using them we obtain very fast and effective algorithms for determining the rigidity of the frameworks [14-16].

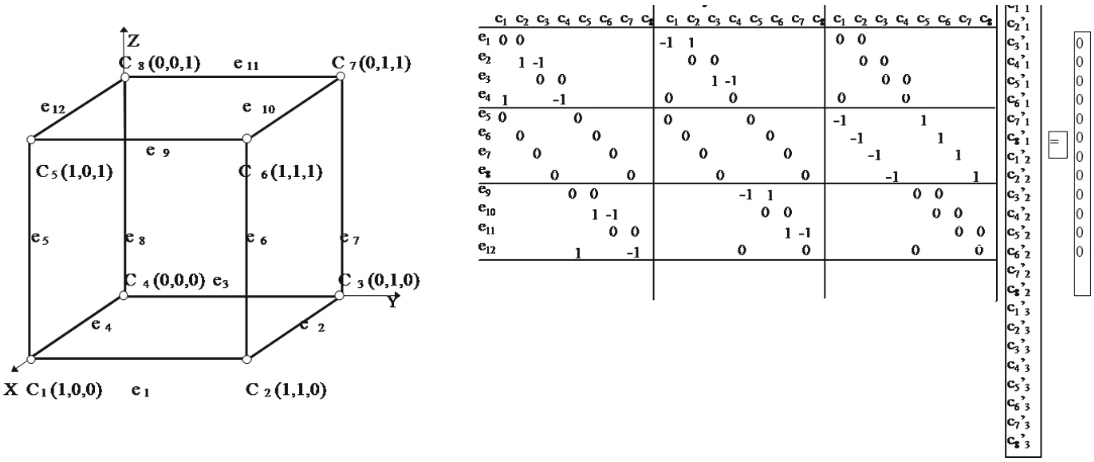

Fig.2. On the left we can see a cubic bar-and-joint framework, which is not a special one, generally the four of the joints of a square faces can move to a skew rhomb. The solution of the system of equation- on the right determines the possible infinitesimal motions.

The rigid body motions of the framework are referred to trivial infinitesimal motions.
Definition 2.2: An F(p) bar-and-joint framework is infinitesimally rigid if it has the trivial infinitesimal motions, only.

2.2 Infinitesimal flexibility of bar-and-joint framework

A framework is called infinitesimally flexible if the system of equation above has a subspace of solutions of more than six dimensions, since the rigid body like motions form a six dimensional subspace. In statics not even infinitesimal motions inside our framework are allowed. E.g. a square with two diagonals as braces is rigid in space but not infinitesimally rigid. Having fixed three joints the fourth joint can move infinitesimally in a direction that is perpendicular to the plane of the square (we disregard that the diagonal elements intersect each other).

2.3 Good characterization of bar-and-joint framework

We can characterize infinitesimal rigidity better than the rigidity of a framework. In [14] Bolker and Crapo gave a graph theoretical model for square grid frameworks. In this paper we give a similar model for zonohedron. Important results for periodic framework bracing with diagonal bars can be found in [16-21].

Some other results [22-27] are also useful to show the wide applicability of the concept and are also connected to our investigations. We note that in an earlier work the planar characterization of rhombic frameworks was solved [10].

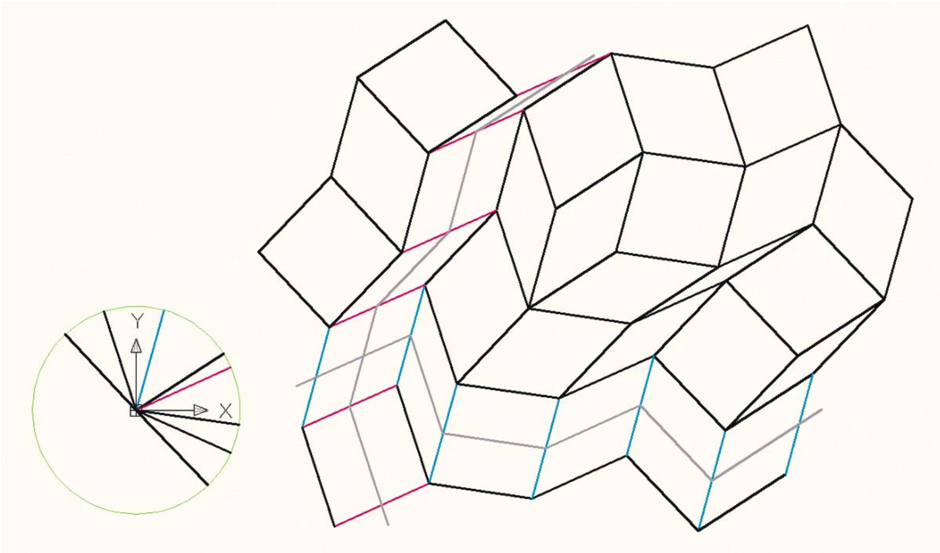

Fig.3. In the middle figure we can distinguish the two equivalent classes of a planar rhombic tiling framework, the parallel grey edges form two equivalent classes, the nearly horizontal one and the nearly vertical one are in different classes. On the left the representing vectors are collected. As generalization, on the right side, we introduce a zonohedron; its horizontal (left-right) edges form the same equivalence class.

3. CHARACTERIZATION OF THE MOTION

3.1 Motion of the rhombic tiling framework

Define equivalence relation between the bars. The opposite bars in the rhombic tiling are equivalent. The bars in the same equivalence class can move only parallel to each other i.e. these bars remain parallel to each other while moving. If we use diagonal braces in such a rhombus, which has sides from two different equivalence classes, then the vectors of the two classes can rotate only together. In Figure 3, in the middle we can observe grey lines. They consist of those segments that are between rods, which are in the same equivalence class. These lines determine some sequences of parallelograms: the so-called de Bruijn lines or Conway worms or ribbons [17].

3.2 Motion of the ZSF

In the ZSF the opposite bars of faces are equivalent. The bars, that are in the same equivalence class can move parallel to each other. The condition of central symmetry does not guarantee planar faces; a skew square is a counter-example. Therefore, to postulate planarity of the faces is important. If we use diagonal braces that form a triangle with two more bars from different classes, then the vectors of the two classes can move only together. The condition of the planarity is not sufficient for the central symmetry, for instant the hexagonal prisms can be counter-example, its regular hexagonal faces can move to a not central symmetrical planar hexagon. Therefore the planarity and central symmetry of the faces are both important.

4. THE RIGIDITY OF THE ZSF

4.1. Auxiliary framework

Define the auxiliary framework to ZSF as follows: the joints of the auxiliary framework correspond to the heads of vectors H_i that represent one of the equivalent classes E_i of the ZSF and let O be the origin of the coordinate system. There are bars between the origin O and heads of vectors H_i, and there are bars H_iH_j between joints H_i and $H_{j,}$ if there exists a diagonal brace between the corresponding equivalent classes of the ZSF.

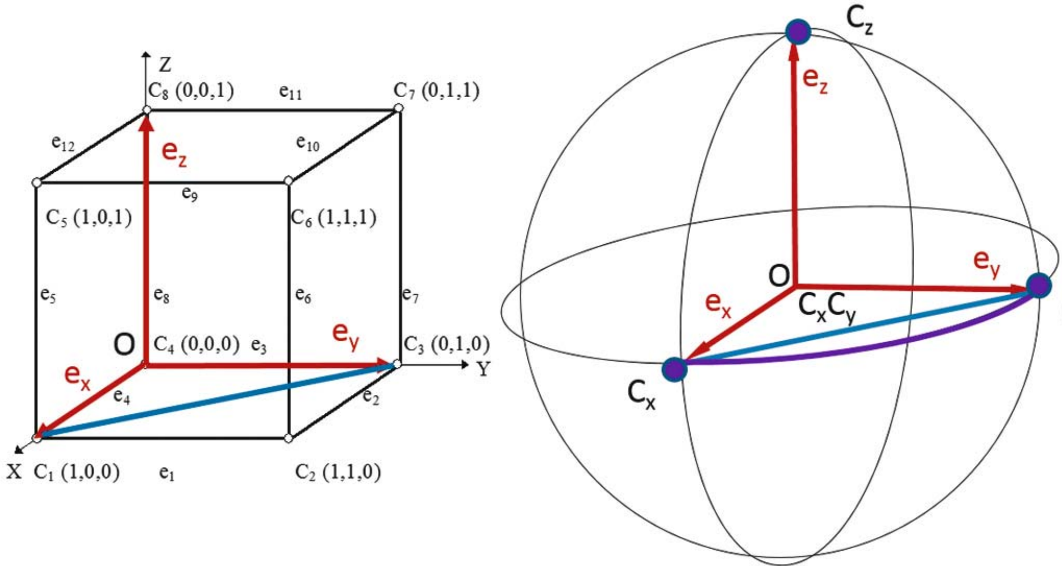

Fig. 4. On the left, a cubic mechanism can be seen that satisfies the planarity and central symmetry conditions having one diagonal brace on the base face. On the right, we can see the corresponding auxiliary framework. The end of base unit vectors and the arc form the rigidity graph G_{ZSF}.

Theorem 4.1: A braced ZSF with parallelogram faces is infinitesimally rigid if and only if the corresponding auxiliary framework is infinitesimally rigid.
Proof: The motion of a bar in the original framework is independent of the motion of the other bars that are in other equivalence classes. This means, if there is no diagonal brace in the framework, then their vectors can rotate around the origin in the space independently from each other. If there is a diagonal brace in the parallelogram corresponding to the two equivalence classes of the sides of the parallelogram if and only if there is brace between the head of the two vectors that characterize the two equivalence classes of the sides of the parallelogram. Hence, the definition of auxiliary framework provides a one-to-one correspondence, disregarding the rigid body like motion, between the motion and infinitesimal motion of the parallelogram sided ZSF and the motion, and the infinitesimal motion of the auxiliary framework.
We can see the auxiliary framework on the right hand side of Fig. 4.

4.2. Characterization of the motion ZSF

We have been given a method to decide flexibility of ZSF with some bracing elements. With it, the description of the exact nature of motions of the braced ZSF with parallelogram faces becomes possible, which is a new family of mechanisms. If we allow faces with more than four sides, then we have to study the planarity of these faces. In this case, the vectors in auxiliary frameworks would not be independent. We plan to investigate this problem in a future work.

4.3. The rigidity of graph G_{ZSF}

The node N_i of the *graph* G_{ZSF} of the ZSF corresponds to the head H_i of the representation vectors. There is an edge N_i N_j between two nodes N_i, N_j of *graph* G_{ZSF} if there is a diagonal brace between the two connected bars that correspond to different equivalent classes of the ZSF. The nodes of G_{ZSF} on Figure 4 are the head of the vectors and the only edge is the purple one. If the vectors of the auxiliary framework are in generic position then the next conjecture holds.
Conjecture: A braced ZSF with parallelogram faces for which the vectors in auxiliary framework are in generic position (algebraically independent over rational numbers) is infinitesimally rigid if and only if the corresponding graph is generically rigid in R^2.

5. A CUBIC ZSF BUILDING IN THE CAD

5.1. The construction for planarity and central symmetry

Firstly, we describe the construction of the more-than four side faces with some further elements in the ZSF that makes them coplanar. Their bars are planar originally, and the opposite bars of the faces are parallel. Then we can subdivide the more-than four sided faces to rhombi or generally parallelograms. The subdivision of polygons (or of the plane) are studied in [27-32] from many aspects.

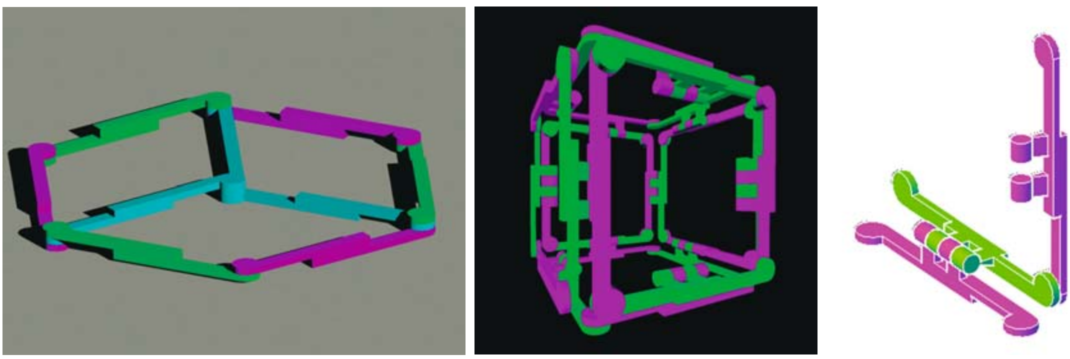

Fig. 5. On the left we introduce the mechanism that preserves the planarity and central symmetry of hexagonal faces. On the middle, we can see a CAD model of the cube ZSF. On the right three adjoining elements can be seen.

The papers [29, 30] deal with edge-to-edge decompositions, or subdivision of a centrally symmetric convex (2k)-gon into centrally symmetric convex pieces.

In our case the edge length of the subdivision is the same as the length of the edges of polygons. A subdivision of a hexagonal framework can be seen in Fig. 5. The three new bars connect every second joint of the six sided polygon face to a common joint, which is placed into the center of the six sides face of the ZSF.

5.2. The construction of the cube ZSF

We give a CAD construction of the cube ZSF based on two simple elements in Figure 5. On the right hand side we can see the two different types of elements that were used to construct the ZSF. Similar construction of the polyhedral frameworks was considered by Laliberté and Gosselin in [11].

6. CONCLUSIONS

In this paper, the flexible ZSF was considered.
We gave a theorem that gives a necessary and sufficient condition for the rigidity of the braced ZSF with parallelogram faces.
A mechanical implementation or model of ZSF could be constructed or printed by 3D printers on the base of a CAD model.

We could answer to the question for the case parallelogram faces ZSF-s, that was asked in [11]: What is the characterization of the mobility or just the infinitesimal motion of the braced (with diagonals), or the non-braced parallelogram faces ZSFs.

Our results are useful for involving the interaction of different structural elements of the systems. The analysis of motions and rigidity of ZSF gives useful input for more complicated three-dimensional structure. Our model should be able to useful the behavior of subframes and their effects on the overall structures. In [33-35] various structural schemes for tall building construction are given. The structures some of them objects, without bracing elements, are regarded as zonohedra.

ACKNOWLEDGMENTS

I would like to thank András Recski for many productive discussions. This research was supported by the National Science Foundation (grant number: 108947.).

NOMENCLATURE

X, Y	joints
XY	bars
p(x)	position of the joint *X*
F(p)	framework
q(x)	*infinitesimal displacement*
Ei	*equivalence classes of equivalent bars*
Hi	*head of the vector that represents the equivalence class Ei*
Ni	*node of the graph GZSF that represents equivalence class Ei*

SUBSCRIPTS

ZSF *Zonohedra with Special Faces*
GZSF *rigidity Graph of ZSF*

REFERENCES

[1] **Cauchy, A. L.,** Sur les polygones et poly`edres, second m´emoire, *Journal de l'Ecole Polytechnique, 19 (1813), 87–98.*
[2] **Dehn, M.,** Uber den Starrheit konvexer Polyeder, *Math. Ann., 77 (1916), 466–473.*
[3] **Legendre, A.-M.,** El´ements de g´eom´etrie. Paris, (an II). *Premi`ere ´edition, note XII (1793), 321–334.*
[4] **Connelly, R.,** The Rigidity of Polyhedral Surfaces, *Mathematics Magazine, Vol. 52 (1979), No. 5, 275–283.*
[5] **Dolbilin, N.P., Shtan'ko, M.A., Shtogrin, M.I.,** Rigidity of zonohedra, *Russian Math. Surveys, 51 (1996), 2, 326–328.*
[6] **Gluck, H.** Almost all simply connected surfaces are rigid, *Lectures Notes in Mathematics, (1975), No. 438, 225–239.*
[7] **Nagy, Gy.,** Diagonal bracing of special cube grids, *Acta Technica Academiae Scientiarum Hungaricae, 106 (1994), 3-4, 256-273.*
[8] **Nagy, Gy.,** The Rigidity of Special D Cube Grids, *Annales Univ. Sci. Budapest, 39 (1996), 107-112.*
[9] **Nagy, G.,** Tessellation-like Rod-joint frameworks, *Annales Univ. Sci, Budapest, 49 (2006), 3-14.*
[10] **Nagy, G., Katona J.,** Connectivity for Rigidity. *Studies of the University of Zilina Mathematical Series, 24 (2010), 1, 6, 59-64.*
[11] **Laliberté, T., Gosselin, M.,** Polyhedra with articulated faces, *12th IFToMM World Congress, Besanc¸on, June 18-21, (2007),*
[12] **Recski, A.,** Matroid Theory and its Applications in Electric Network Theory and in Statics., Akadémiai Kiadó, Budapest and Springer, Berlin, (1989),
[13] **Roth, B.,** Rigid and Flexible Frameworks, *The American Mathematical Monthly, Vol. 88 (1981), No. 1, pp. 6–21.*
[14] **Bolker, E., Crapo, H.,** Bracing rectangular frameworks I., *SIAM J. Appl. Math. 36 (1979), 3 473-490.*
[15] **Baglivo, J.A., Graver, J.E.,** Incidence and Symmetry in Design and Architecture, Cambridge University Press, Cambridge (1983).
[16] **Radics, N., Recski, A.,** Applications of combinatorics to statics – rigidity of grids, *Discrete Applied Math. 123 (2002), 473-485.*
[17] **Frettlöh, D., Harriss, E.,** Parallelogram tilings, worms and finite orientations, *Discrete and Computational Geometry (2013), 4, 9531-539*
[18] **Friedman, N., Ibrahimbegovic, A.,** Overview Of Highly Flexible, Deployable Lattice Structures Used In Architecture And Civil Engineering Undergoing Large Displacements, *Ybl Journal Of Built Environment, 1, (2013), 1, 85-103.*
[19] **Gatt, R., Mizzi, L., Azzopardi, J. I., Azzopardi, K. M., Attard, D., Casha, A., Briffa, J., Grima, J. N.,** Hierarchical Auxetic Mechanical Metamaterials, *Nature, 25 (2013), 36, 5044.*
[20] **Guest, S. D., Hutchinson, J. W.,** On the determinacy of repetitive structures. *J. Mech. Phys.Solids., (2003), 51, 383–391.*
[21] **Power, S. C.,** Crystal Frameworks, Symmetry and Affinely Periodic Flexes, *New York J. Math., 20, (2014), 1–29.*
[22] **Jordán, T, Domokos, G, Tóth, K.,** Geometric Sensitivity of Rigid Graphs, *SIAM Journal on Discrete Mathematics, 27 (2013), 4,1710-1726.*
[23] **Kapko, V., Treacy, M. M. J, Thorpe, M. F., Guest S. D.,** On the collapse of locally isostatic networks. *Proc. R. Soc. A., (2009), 465, 3517–3530.*
[24] **Pellegrino, Z. S.,** Foldable bar structures. *Int. J. Solids Struct., 34 (1997), 1825–1847.*
[25] **Friedman, N., Farkas, GY., Ibrahimbegovic, A.,** Deployable/Retractable Structures Towards Sustainable Development, *Pollack Periodica, Vol. 6. (2011) No. 2, 85-97,*
[26] **Kovács, F., Tarnai, T., Guest., S.D., Fowler, P.W.,** Double-link expandohedra: a mechanical model for expansion of a virus, *Proc. R. Soc. A., 460 (2004). 3191-3202.*
[27] **Tanaka H., Shibutani, Y., Izumi, S., Sakai, S.,** Planar mobility modes of eight-barjointed structures with a single degree of freedom. *Int. J. Solids Struct., 49 (2012), 1712–1722.*
[28] **Bölcskei, A. - Molnár, E.,** How To Design Nice Tilings? *KoG, Scientific and Professional Information Journal of the Croatian Society for Constructive Geometry and Computer Graphic 3 (1998), 21–28.*
[29] **Böröczky, K.J., Réti, T., Wintsche, G.,** On the combinatorial characterization of quasichrystals, *Journal of Geometry and Physics, 57 (2006) 39-52.*
[30] **Frittmann, J., Lángi, Zs.,** Decompositions of a polygon into centrally symmetric pieces *arXiv:1504.05418*
[31] **Horváth, A. G.,** Dissections of a centrally symmetric hexagon, *Bolyai Soc. Math. Stud. 6 (1997), 327-334.*
[32] **Reti, T., Böröczky, K.J.:** Topological Characterization of Cellular Structures, *Acta Polytechnica Hungarica, 1 (2004) p. 59-85.*
[33] **Richard Liew, J.Y.; Balendra, T., Chen, W.F.** "Multistory Frame Structures" Structural Engineering Handbook Ed. Chen Wai-Fah Boca Raton: CRC Press LLC, 1999
[34] **Zalka, K.A.,** , A simple method for the deflection analysis of tall wall-frame building structures under horizontal load. *The Structural Design of Tall and Special Buildings, 18, (2009) No. 3, 291–311.*
[35] **Zalka, K.A., Armer, G.S.T.,** 1992, Stability of large structures, (Oxford: Butterworth-Heinemann).