PEDAGOGICAL INVERSION

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Pedagogical inversion allows connecting disciplines, which are otherwise considered insoluble. This paper presents one exemplary implementation of the pedagogical inversion in three undergraduate courses at Sofia University. The practical application reaches topics from other disciplines and repacks traditional problems from new and unexpected points of view like the one that wrong solutions to problems are at least as educational as the correct solutions. Finally, the paper presents some of the educational materials that are used in these courses. These materials range from power-point presentations to libraries of hundreds of programming examples and a collection of multimedia animations and mathematical movies.

Keywords: pedagogical inversion, reusing mistakes, learning through restriction, pandisciplinarity

Pedagogical inversion

Pedagogy can be considered to be both science and art. According to Merriam-Webster Dictionary, pedagogy is defined as "the art, science, or profession of teaching; especially: education" (Pedagogy, 2011). The etymological roots of pedagogy can be traced back to Ancient Greece where "paidagogos" were the people who led children to school ("paidos" means "child" and "ago" means "lead").

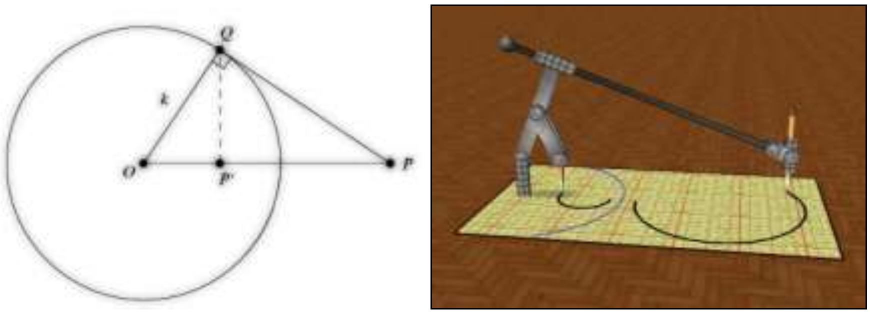
As a science the pedagogy relies on a set of scientific methods to achieve predictable results. As an art it provides freedom for experiments by using the expressive power of intangible artifacts. Additionally, like art, pedagogy is a subject of interpretation and a consequent re-interpretation. The first interpretation in art is done by the artist and the viewer/listener re-interprets it. In education, the first interpretation is done by the teacher, and the second one is done by the students. The artistic nature or pedagogy postulates that both interpretations might disagree with each other. However, the scientific nature of pedagogy tries to avoid any mismatching.

This paper is not about the pedagogy in its traditional meaning and application. The focus is on an approach that uses, at first sight, ideas that ruin the mainstream approach of applying pedagogy. Due to the lack or proper terminology, this approach is referred to as *pedagogical inversion*.

The term *inversion* is well defined in a broad spectrum of knowledge areas like Mathematics, Literature, Physics, Genetics, and Chemistry.

In mathematics, for example, geometrical inversion it is a mapping of the points on a plane. Traditionally, it is represented in a flat 2D diagram (see Figure 1, left), which describes the construction of an inversion. This representation is rather abstract, as it does not produce an intuitive appreciation of how inversion works.

Figure 1. *Traditional 2D representation of geometrical inversion and a 3D model of a virtual device, which implements this transformation*



The right image in Figure 1 is a snapshot of a 3D virtual model (Boytchev, 2010a) of a mechanism that implements geometrical inversion. If one of the pencils draws an image, the other one draws its inverse image. And it does not matter which pencil of the two is picked as a main pencil. Inversion is completely symmetrical in this respect.

In an educational context the inversion represents the projection of several disciplines within the scope of another one. We name this representation a *pedagogical inversion*, to distinguish it from both multi- and interdisciplinary approaches, which also cross-disciplinary.

Traditionally, the curriculum is granulated into disciplines and cross-disciplinary topics bridge only areas that are more or less conceptually close. The pedagogical inversion by itself does not bridge disciplines, as it is done in multi- and interdisciplinary approaches. Instead it maps them in a way that mapping could occur between disciplines, which are otherwise considered "insoluble".

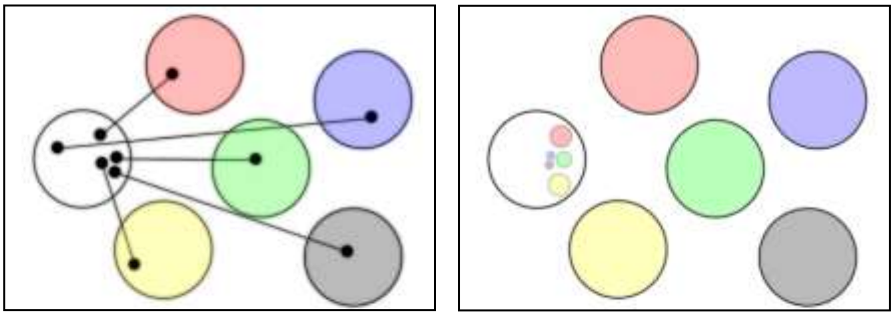
Some educators may treat such pedagogical inversion as pedagogical diversion. The rest of the paper presents one exemplary implementation of the pedagogical inversion in three undergraduate courses at Sofia University: "*Educational Languages and Environments*", "*Computer Graphics*" and "*Geometry of Motion*".

Grounds for introducing pedagogical inversion

When teaching is considered just as a duty, it is also treated in a commercial way. There is a consignment of goods (things to learn) that must be "given" clients (students) in order to get some payment. The monotonous repetition of this cycle throughout many years leads to a kind of accustoming in teaching and of teaching. The same happens with some of university students. During lectures they are in "read only" mode and they switch into "write only" mode only when preparing homework or during exams.

The author's personal observation throughout the last five years of teaching at university level is that every discipline is considered and treated in isolation. It is like students have separate "heads" for different disciplines. In a Computer Graphics lecture, they do not remember anything from Geometry, however back in a geometry lecture they suddenly recall all geometric concepts (at the cost of forgetting all about Computer Graphics).

Figure 2. *Bridging disciplines (left) versus projecting disciplines (right)*



This observation is shared by many educators and a conventional “remedy” for this situation is believed to be provided by inter- and multidiscplinarity. These approaches are not new and they have a very positive impact. However, they are usually applied only to disciplines that are close to each other.

Figure 2, left, represents connections between the left-most discipline to other disciplines. The right image in the figure shows the same sets of disciplines, but now they are mapped into the left-most one. Similarly to the geometrical inversion, everything outside the discipline-circle could be mapped into something inside the discipline-circle.

The main benefit of the mapping in the pedagogical inversion is that it helps to merge disciplines which are considered “insoluble”. The main disadvantage is that it is much harder to implement such mapping compared to the already traditional inter- and multidisciplinary approaches.

The practical application of pedagogical inversion is a collection of various techniques, which focus on several ideas: sharp jumps from one topic to another topic in another discipline; using mistakes as educational resources; breaking (or at least disregarding) they ways problems are presented and their solutions are expected; and learning through restrictions.

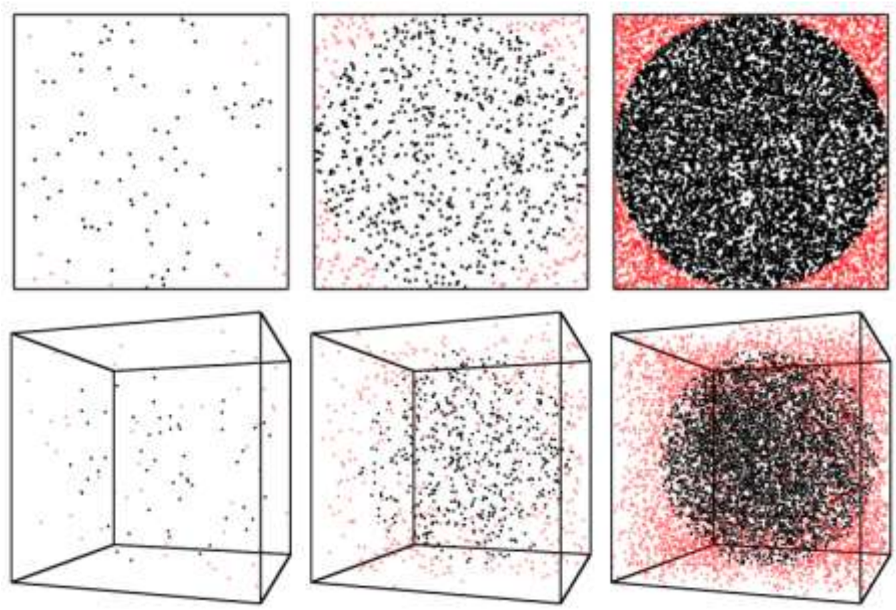
Example 1: Points in a sphere problem

One of the problems, illustrated in the Computer Graphics course, is the generation of random points in a given shape (Boytchev, 2007). The solution exercises several widely used concepts – dynamical naming of objects, interval arithmetic, and primary object collision detection.

The first problems are easy – generating N random points in a circle. One approach is to generate Cartesian coordinates of points in a square and filter out the ones that are outside the circle. Another approach is to use a generative equation that generates points directly inside the circle. A third approach is to use spherical coordinates. Figure 3 (upper row) shows snapshots of an exemplary application generating 100, 1000 and 10000 random points in a circle using Cartesian coordinates.

When this problem is solved, another one is given – generating N random points in a sphere, see Figure 3 (bottom row). The same approaches can be reapplied, but this time the solutions are used to provide pedagogical inversion in two aspects: utilizing the educational potential of mistakes and following educational meanders to other disciplines.

Figure 3. *Random points in a circle and a sphere (100, 1000 and 1000 points)*



At the beginning the students are notified, that generating *N* points in cube and then filtering some of them will not create *N* points in the sphere. Thus, the initial solution is wrong. Of course, we can loop point generation and count only the points, which are in the sphere, however the initial wrong solution provides ideas for a new development.

Students are asked to estimate intuitively the amount of points in the cube that are left outside the sphere. This challenge to students' intuition was given for 5 consecutive years and the majority of guesses are that between 10% and 30% of points are dropped out. It appears that this interval is adequate for the 2D case, but is completely off-scale for the 3D situation.

The program that generated random points is then modified to count the points. Because of the random nature of generation, the number of points outside the sphere varies even if *N* is fixed, but the ratio oscillates around specific values.

Students are usually surprised to see that almost half of the points in the cube remain outside the sphere in that cube. Table 1 shows the percentage of dropped points for the snapshots in Figure 3.

Table 1. *Dropped out points for a circle and a sphere. The estimated values of π are based on the measurements for spheres*

	Circle	**Sphere**	**π**
100 points	19.00%	46.00%	3.2400
1000 points	20.30%	45.90%	3.2460
10000 points	21.36%	47.44%	3.1536
Surface (volume) ratio	1-π/4 ≈ 21.46%	1-π/6 ≈ 47.64%	π

To find a reasonable explanation of these unexpected results, students are asked to calculate the probability of a random point being outside the sphere. To solve this task, students should use knowledge from two other courses – Analytical Geometry and Probabilities and Statistics. It appears that the probability can be expressed by the volumes of the cube (V_C) and the sphere (V_S). Thus, the probability can be calculated as:

$$P = 1-V_S/V_C = 1- (4/3)\pi R^3/(2R)^3 = 1-\pi/6 \approx 0.4764$$

Apart from the formal proof about the expected number of dropped out points, the solution shows a way for experimental estimation of π:

$$\pi \approx 6 * (1 - DroppedPoints/AllPoints)$$

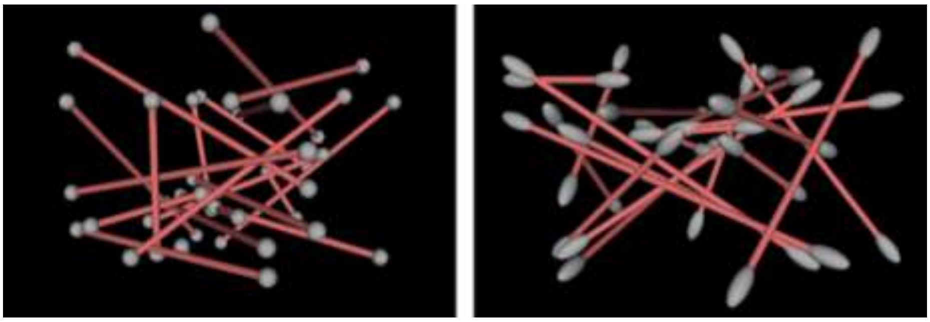
The last column in Table 1 shows the estimated values of π for different number of points in a cube. Naturally, when the number of points increases, the estimation become more accurate.

Calculating volumes of geometrical shapes and estimation of π are off-topics for Computer Graphics course. Nevertheless, they provide a valuable lesson to students that disciplines are mutually bound. Things learned in one course could be used in another. A cross-discipline knowledge application is an indication of a true understanding of various concepts.

Apparently, this problem is a valuable lesson to teachers too. It shows that mistakes could be a source of new explorations. Mistakes have a huge, yet underexplored, educational value. If people do learn from their mistakes, then why mistakes are massively being avoided in the educational process? A very essential set of competencies is to deal effectively with mistakes. Such competencies have a very practical application in people's lives.

Teachers do not like mistakes. It is hard to resolve mistakes in real time and in front of all the students. Also, some teachers assume that mistakes ruin their authority and prestige. Yet, dealing with mistakes is a priceless lesson and teachers must not be afraid of mistakes.

Figure 4. *Converting ball-and-stick models into ear-sticks*



Example 2: Redressing problems

Many problems given in disciplines related to Mathematics and Computer Science are "boring". A component of the pedagogical inversion is to dress problems in a way that makes them intriguing.

Keeping students' interest is a key factor for successful education. This section shows how problems are changed without loosing their educational values. The problems are:

- Modeling ear-sticks
- Modeling flies in a water closet
- Modeling a platonic love
- Modeling a cockroach crawling on a lamp
- Modeling a samurai splitting a sausage in three

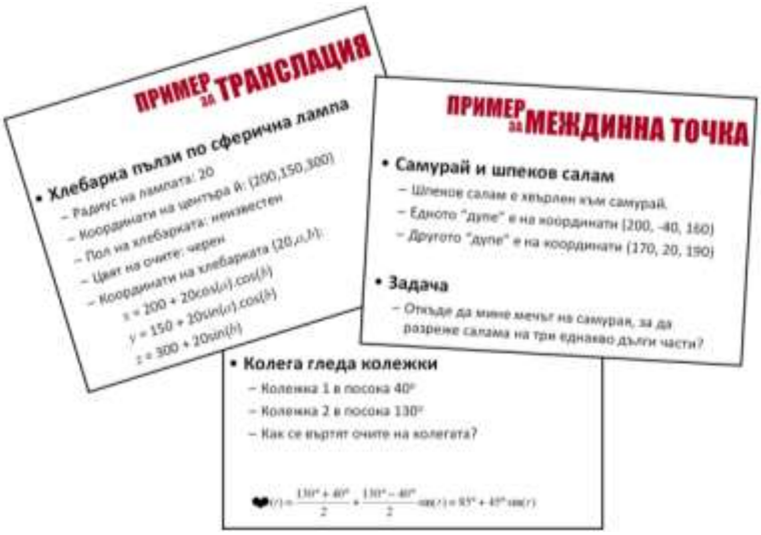
The visualization of graphical primitives provides freedom to students' artistic vision. For example, instead of showing a group of segments drawn as straight lines connecting two points, it is possible to make them more real, by replacing the segments lines by cylinders and attaching spheres at both ends. This would make models closer to the ball-and-stick models used in Chemistry courses (Figure 4, left).

The ball-and-stick model can be further modified to model ear-sticks as shown in Figure 4, right. Although straightforward, converting spheres into ellipsoids requires some additional thinking. Spheres are invariant to orientation in space, while ellipsoids should be aligned along the sticks.

The modeling of flies flying in a country-side water closet is very intriguing. Students have never expected to solve problems related to flies and water closets in university. However, there is a lot of mathematics embedded in this problem. One solution is to use separate asynchronous harmonic motions along *X*, *Y* and *Z* axes – a 3D variation of the Lissajous curve (Weisstein, 2011). The same technique is used in the Predatory Plants model in Figure 5, where flying insects follow trajectories defined parametrically as 3D Lissajous curves (Boytchev, 2011b).

Figure 5. *Using 3D Lossajous curves for insects' trajectories*

Figure 6. *Three problems about cockroaches, love, samurais and sausages*



Snapshots of the lecture presentation describing the three other problems are shown in Figure 6. The top-left problem is: A cockroach crawls on a spherical lamp. The lamp radius is 20, the center is at (200,150,30). Express the trajectory of the cockroach in polar and in Cartesian coordinates.

The bottom problem of Figure 6 is about modeling male student rolling eyes towards two female students that are in directions 40^{O} and 130^{O}.

The top-right problem of Figure 6 is about a sausage thrown at a samurai. The coordinates of the sausage's ends in 3D space are (200,-40,160) and (170,20,190). The problem is to find the coordinates, where the samurai's sword will slice the sausage in three equally long parts.

As described earlier, wrapping problems with interesting and unexpected stories increases the students' interest and curiosity. Moreover, students are unaware what the next problem would be; they are tossed in different, often incompatible, directions. The students' reactions are quite positive. The next dialogue is from the course Educational Languages and Environments during the winter semester of 2010/2011 academic year:

Students: *How do you manage to invent such problems?*
Lecturer: *I've graduated in this faculty*
Students: (LOL)
Students: *Will we become the same?*
Lecturer: *Only if you graduate*
Students: (ROFL)

Example 3: Breaking the patterns

People have a tendency to automate their actions in order to reduce cognitive load and stress. The automation in this case refers to mechanical repetitions of the same actions or processes over and over. Automation of repetitive actions is good, until it starts to affect negatively human creativity and inspiration.

Patterns have occupied teaching. This is most obvious when a given topic is presented in the same way year after year. Unfortunately, patterns have occupied learning too. One of the possible solutions to this problem, as suggested by the pedagogical inversion, is to provide information, which is not fully "digested". This will force the students to "process" it before they can use it to learn.

An intriguing experiment was carried out in 2010. The lectures of the course Educational Languages and Environments are in Bulgarian language. However, the lecture on November 12th was presented in Macedonian – see Figure 7.

The official explanation to students, given before the beginning of the lecture, was that there was one Macedonian student enrolled in the course, thus one of the lectures will be in her native language.

Figure 7. *Snapshots from the presentation in a foreign language*

Macedonian and Bulgarian languages are very close, so students could relatively easily understand the meaning of the slides; however, they were surprised that all slides were in Macedonian. They thought that it was a joke.

During the whole lecture all students paid close attention to all shown materials, translating texts and discussing in the background possible interpretations of some strange words. Some of the students that came after the announcement stood at the doors in disbelief. They were in the correct room, at the correct time, the lecturer was the right one, the students in the room were the right ones… but the presentations were different.

Linguistic challenges are spread all over the course. In one lecture, there was a discussion about forming plural forms of nouns in Bulgarian language. Several different forms were mentioned but none of them was applicable for the plural of “Winnie-the-Pooh”.

Similarly, every year the students are asked to find the word in Bulgarian that means the stiff connections finishing off the shoelaces. The Bulgarian translation of “aglet” is very unpopular; so many people do not know it. At last, the 2010-year students managed to provide an answer.

Breaking the patterns could be applied in a stricter educational context, without spreading out into various topics. For example, one of the typical patterns for teachers in Mathematics or Computer Science is to use common names for objects and variables. Why should triangles have vertices *A*, *B* and *C*? Why should index variables for loops be *i* and *j*? Why should axis *Z* point upwards?

By presenting entities with various names, students feel better, that names are relative and subjective. Of course, proper naming would ease the understanding, but sometimes relying on conventional names could enforce patterns that make understanding harder. A typical example is a circle with equation $x^2+y^2=r^2$ that must be oriented so that it lies in plane *OXZ*. The new equation is (obviously) $x^2+z^2=r^2$, but many students have problems applying the “template” of a circle equation in a more abstract way.

Figure 8 illustrates a snapshot of the lecture about the geometrical bases of computer graphics. The figure shows a linear combination of points that are named with the Cyrillic letters *Ш*, *Ю* and *Ъ*.

Figure 8. *Using unconventional alphabet for points' names*

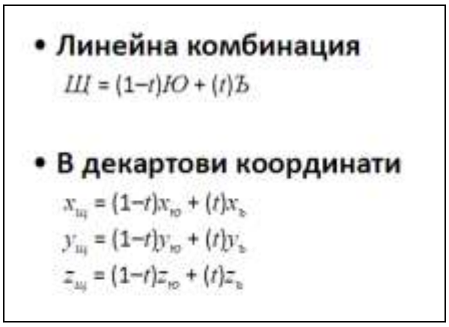
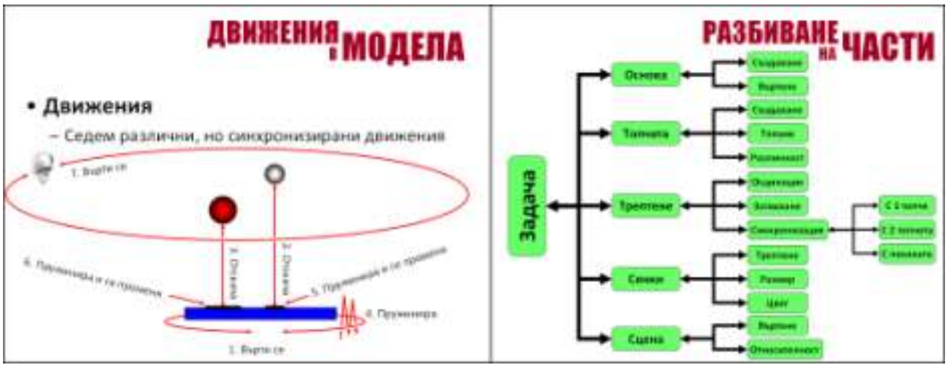


Figure 9. *Description of the Bouncing balls problem*



Example 4: Learning through restrictions

Development without restrictions is a wide-spread and highly-praised concept. It is really something that makes learning easier, but not necessarily deeper. Pedagogical inversion addresses this issue by promoting an "opposite" approach – learning through restrictions. The general concept is that handling restrictions is much closer to real-life situations, than working in an unrestricted environment. Imagine someone who learns to do a somersault in a gymnastic hall. When he is ready with it, let him do it in a tiny closet. Although this is rather distant analogy, it correctly represents one of the approaches of pedagogical inversion. Infinite possibilities and reaching new horizons are something that is not in the primary focus, they are left for the conventional pedagogical methods.

A problem with the bouncing balls will be used to illustrate three of the many types of restrictions that can be utilized. The problem is to model two balls bouncing on a vibrating platform. The model should support 7 distinct motions. A snapshot of the slide describing the motions is shown in Figure 9, left. The problem is solved by dividing it into simpler problems by generating a tree which leaves represent simple problems – Figure 9, right.

The first restriction that is imposed on the solution is to use only the function sine (or cosine) as a base function for all motions. Rotational motions and vibrations are easily constructed by using such periodic functions, but falling on the platform and bouncing off it is something that is related to ballistic (parabolic) curves, not to trigonometric curves.

Modeling of a bouncing controlled by gravity seams to be unsolvable, however, the students are presented with a working model as a proof that there is a solution. And the solution is to use part of the sine curve instead of a parabola. Although mathematically and mechanically incorrect, this solution provides visually adequate motion.

The lesson learned by this example is that we can introduce imperfections up to the level of imperfection of the “weakest” node. The human perception of bouncing is the least accurate in the chain program-animation-viewer, so we can “disaccurate” (sic) the model as much as it is still unnoticeable by people.

The second restriction is to make variations of the model with the least amount of source code changes. The original solution features bouncing balls as seen by an external viewer. The new problems are to model the viewer is an ant standing on one of the balls; or a mosquito trying to alight on a ball. The solution of these problems requires the change of only one line in the source code of the program. All motions and object behavior are kept the same. This is a nice representation of the concept, that even if the scene is dramatically different, the mathematical backbone of all motions in it is practically the same. All the difference is just a matter of a view point.

The third type of restriction is related to one specific problem that occurs during the vibration of the platform. Initially, the amplitude of vibration is modeled by a sine function, but when a ball hits the platform, it should first move downwards.

Figure 10. *Possible fixing of the vibration direction*

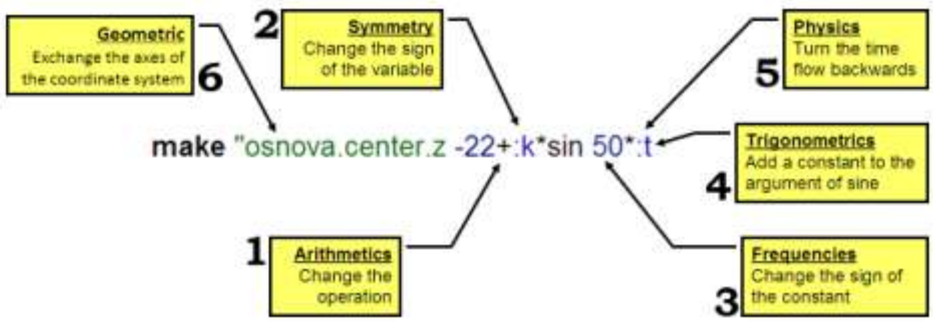


Figure 10 shows the statement that must be fixed. The dialogue, which is done with the students, tries to provoke them to find as many solutions as it is possible. Usually, students’ first suggestion is pure arithmetical – to change the operation in the expression (see block 1). What if it is forbidden to change this sign? Then another solution is to change the sign of the amplitude, which is stored in variable *k* (see block 2). This is also a correct solution, but what if *k* is used in other parts of the program and it cannot be changed? A third solution is to think physically – we can negate the frequency by using -50 instead of 50 (see block 3). If this is not possible, then we can use rules from trigonometry – we can add a constant to sine so as to shift it along the time axis and make it go downwards (see block 4). What if this is not possible too? Then we can go to a more abstract level, changing the direction flow of the time *t* (see block 5). If time starts from 0, but goes in direction -1, -2, -3… then the sine function will be decreasing. And finally, what if the time should not be modified? Then we can modify the space of the continuum turning the up direction downwards. We can apply this transformation locally, affecting only the platform.

All these solution to the same problem are mathematically equivalent. When they are expressed mathematically, we would get the same equation. However, these six solutions illustrate six distinct view points on the same problem, such as each of the view point leads to a specific interpretation of the solution.

If this problem were presented in a conventional pedagogical style, it would be almost impossible to show the diversity of possible solutions and their interpretations. Within the pedagogical inversion, posing restrictions on what resources or ideas to use, students are forced to explore a much larger spectrum of options, something which would be quite helpful for them in other situations.

Example 5: Pandisciplinarity

The final example of how pedagogical inversion is applied is by the use of pandisciplinary materials. Disciplinarity in educational context has evolved from mono- through multi- and up to interdisciplinarity. The next natural phase is pandiscplinarity (Boytchev, 2011a), which would allow the use of a single educational material in different, often incompatible disciplines.

Figure 11. *Virtual models built by students*

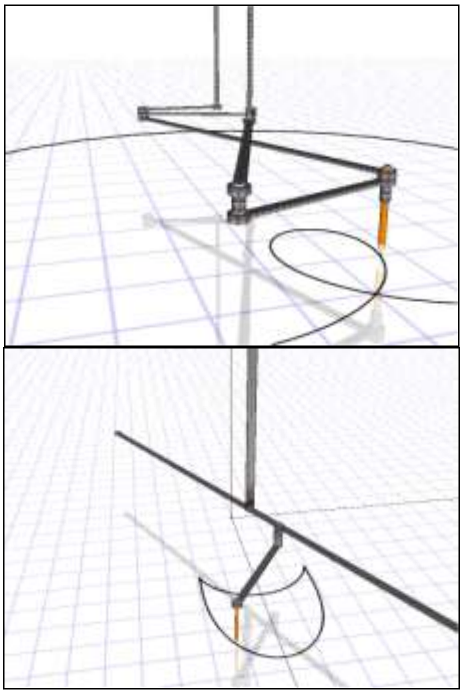


Figure 12. *Some of the devices in the collection of virtual mechanical devices*

The course Geometry of Motion is a blended course covering topics from Geometry and Mechanics. During the spring semester of 2010/2011 academic year, students enrolled in this course were given for the first time the option to build 3D virtual models of devices that draw mathematical curves.

Although the course did not cover any programming or computer graphics topics, volunteering students could try to apply together knowledge of several disciplines. More over, the programming environment and the programming language were totally new to them; and they had to learn them with minimal assistance. Two models, built by students, are shown in Figure 11.

The construction of these models uses a software library which making was inspired by a collection of other virtual devices (Boytchev, Sendova, & Kovatcheva, 2011). Snapshots of the devices in the collection are shown in Figure 12. The devices are built in the same programming language that is used in the courses referenced in this paper.

The collection of devices also inspired the production of a movie (Boytchev, 2010b), which is still to unroll its full pandisciplinary potential – the movie can be used in courses related to Geometry, Computer Science, Mechanics, Art, Film production, History of sciences.

The movie features seven different methods for generation of ellipses. Figure 13 represents four of them. The movie is done programmatically – i.e. there is source code that generates animations frame by frame.

Conclusion

This paper presented some of the main concepts of pedagogical inversion, which is not a unified and monolithic approach, but a collection of various approaches. These approaches are not widely used, because they require a lot of efforts in the initial preparation of the educational materials.

Figure 13. *Snapshots from the movie "Ellipses…"*

Also, some of the approaches might be considered as conflicting with the mainstream pedagogical practices, like the intentional introduction of mistakes and imposing of restriction.

The paper also described examples of five approaches of the pedagogical inversion that have been applied in a classroom setting: (1) extracting experience from wrong solutions; (2) the redressing of problems without scarifying their educational value; (3) breaking the patterns in teaching and learning; (4) using various types of restrictions to increase the impact of learning; and (5) utilizing pandiscipinary aspects of educational resources.

Some of the future research activities in the area of the pedagogical inversion would be to identify more approaches, which are fruitful, but yet neglected; as well as continuing the application of the approaches described in this paper, so that to collect more observations.

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