# HOLISTIC APPROACH TO THE TEACHING OF MATHEMATICS

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*The pure approach of mathematical studies for example a theorem in itself, is not especially interesting for pupils. The holistic approach of mathematics in more studies seems to be interesting. Considering the mathematical methods used in other areas, we can find different possibilities for mathematical applications already in the primary (or secondary) school.*

**Keywords:** Mathematical methods used in language, history, geography, astronomy and mathematics in itself

The aims of holistic approach in the teaching of Mathematics are:

- To show the usefulness of Mathematics. (Mathematics in itself is not interesting for everybody.)
- Understandable mathematical applications,
- Mathematical methods used in other areas:

*Language*: Let us consider the proverb „Early bird catches the worm." The first task is to elaborate the exact meaning of this sentence," The bird is early and it catches the the worm." or „If the bird is early then it catches the worm." The genuelly exiting problem is the negation of the proverb. Pupils give a wide range of answers. Which is the correct one?

*History*: Comparing the hierographical number system with the signs of numbers used in Mesopotamia in Ancient Times, the question is: Which is the better system? Why?

*Fine arts*: What does the notion „gold section" mean?

*Music*: How the sound scale „c,d,e,f,g,a,h, (upper)c" constructed in about the century 6th B.C.?

*Geography*: Measurement of the earth's circumference in the century 3th B.C.

*Astronomy*: Measurement of the moon's diameter in the century 3th B.C.

*Mathematics in itself*: Our calculator with 10 characters, says that $\pi = 3{,}141592265$ , but we know, that this is an approximate result, only. Using the same calculator let us give the 11th character, too!

Problems and questions mentioned above are in connecting with mathematical logic, numerical basic – operations (addition, multiplication, power), geometry, fractions and simple computations, respectively.

## Mathematics in Language

Both Language and Mathematics are special expressions of human thinking. Language is wider and richer, Mathematics is narrow but more precise. We demonstrate it by the following example. Considering the proverb „Early bird catches the worm." we set up the problem of its precise negation. In general we get three variants of answers:

I. variant: Late bird catches no worm.
II. variant: Late bird catches the worm.
III. variant: Early bird catches no worm.

Mathematics accepts only one answer from among the above. Which one is it?

Let us begin the precise meaning of the proverb „Early bird catches the worm." ! We can choose between two versions:

- First version: The bird is early and it catches the worm.
- Second version: If the bird is early then it catches the worm.

We assume that the second version is chosen (by the majority of pupils).

In the next we give a mathematical forms of sentences and connectives.

- Judgement „The bird is early." is denoted by „p".
- Judgement „It catches the worm." is denoted by „q".
- Connective „and" is denoted by $\wedge$ .
- Connectives „If….then" is denoted by $\Rightarrow$.

Connectives are mathematical operations between judgemens. So,

- First version: $p \wedge q$ (Conjunction.)
- Second version: $p \Rightarrow q$ (Implication.)

Both judgemens p and q have two possibilities: either true (denoted by „t") or false (denoted by „f"). So, for the conjunction and implication we have, respectively:

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| t | t | t |
| t | f | f |
| f | t | f |
| f | f | f |

and

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| t | t | t |
| t | f | f |
| f | t | t |
| f | f | t |

.

The negation is denoted by „$\neg$", that is

| $p$ | $\neg p$ |
|---|---|
| t | f |
| f | t |

.

Using our assumption the (mathematically precise) negation of „Early bird catches the worm." is

$$\neg(p \Rightarrow q).$$

Considerings the variants and versions mentioned above, we may have six cases:

Case 1. $\neg p \wedge \neg q$ (The bird is late and it does not catch any worm . See I. variant.)

Case 2. $\neg p \wedge q$ (The bird is late and it catches the worm. See II. variant.)

Case 3. $p \wedge \neg q$ (The bird is early and it does not catch any worm. See III. variant.)

Case 4. $\neg p \Rightarrow \neg q$ (If the bird is late then it does not catch any worm .)

Case 5. $\neg p \Rightarrow q$ (If the bird is late then it catches the worm.)

Case 6. $p \Rightarrow \neg q$ (If the bird is early then it does not catch any worm.)

Among 1-6. are there any cases having the same meaning? This question seems to be complicated. We already need the mathematical methods. First af all we devise the description of the precise negation $\neg(p \Rightarrow q)$.

| p | q | $\neg(p \Rightarrow q)$ |
|---|---|---|
| t | t | f |
| t | f | t |
| f | t | f |
| f | f | f |

The correct cases have this description, exactly. Let us see the descriptions of Cases 1-6!

Ad Case 1.

| p | q | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|
| t | t | f | f | f |
| t | f | f | t | f (Error!) |
| f | t | | | |
| f | f | | | |

Ad Case 2.

| p | q | $\neg p$ | $\neg p \wedge q$ |
|---|---|---|---|
| t | t | f | f |
| t | f | f | f (Error!) |
| f | t | | |
| f | f | | |

Ad Case 3.

| p | q | $\neg q$ | $p \wedge \neg q$ |
|---|---|---|---|
| t | t | f | f |
| t | f | t | t |
| f | t | f | f |
| f | f | t | f |

ONE OF WINNERS

Ad Case 4.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $\neg p \Rightarrow \neg q$ |
|---|---|---|---|---|
| t | t | f | f | t (Error!) |
| t | f | | | |
| f | t | | | |
| f | f | | | |

Ad Case 5.

| $p$ | $q$ | $\neg p$ | $\neg p \Rightarrow q$ |
|---|---|---|---|
| t | t | f | t (Error!) |
| t | f | | |
| f | t | | |
| f | f | | |

Ad Case 6.

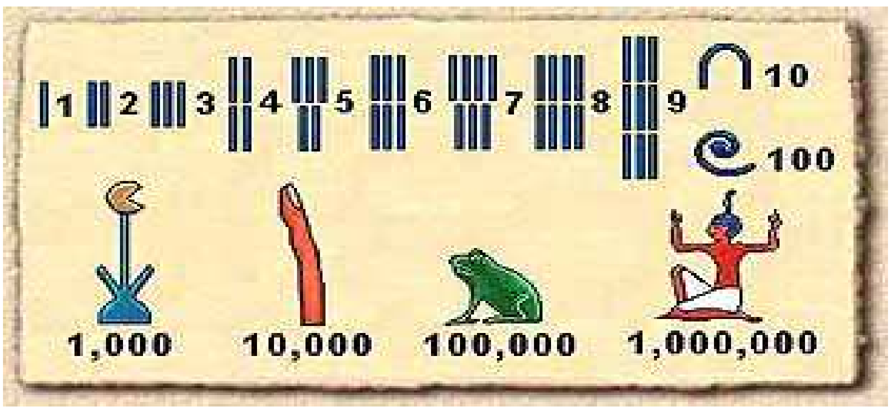
| $p$ | $q$ | $\neg q$ | $p \Rightarrow \neg q$ |
|---|---|---|---|
| t | t | f | f |
| t | f | t | t |
| f | t | f | t (Error!) |
| f | f | | |

Hence, we can see that the Case 3. (III. variant) is correct, alone.

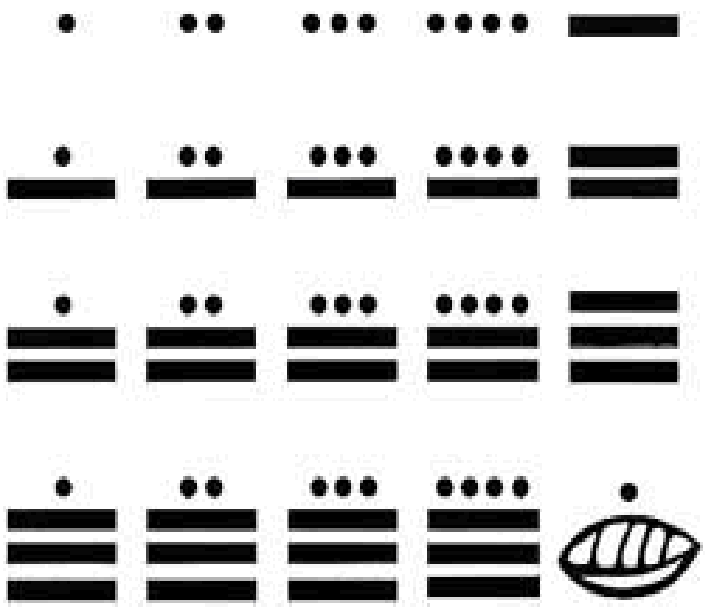
## Mathematics in History

Historia est magistra vitae. (History is the master of life.) To follow the way of history is very fruitful at school, too. For example to show the development of concept of number is very useful through ancient egyptian hieroglyps.

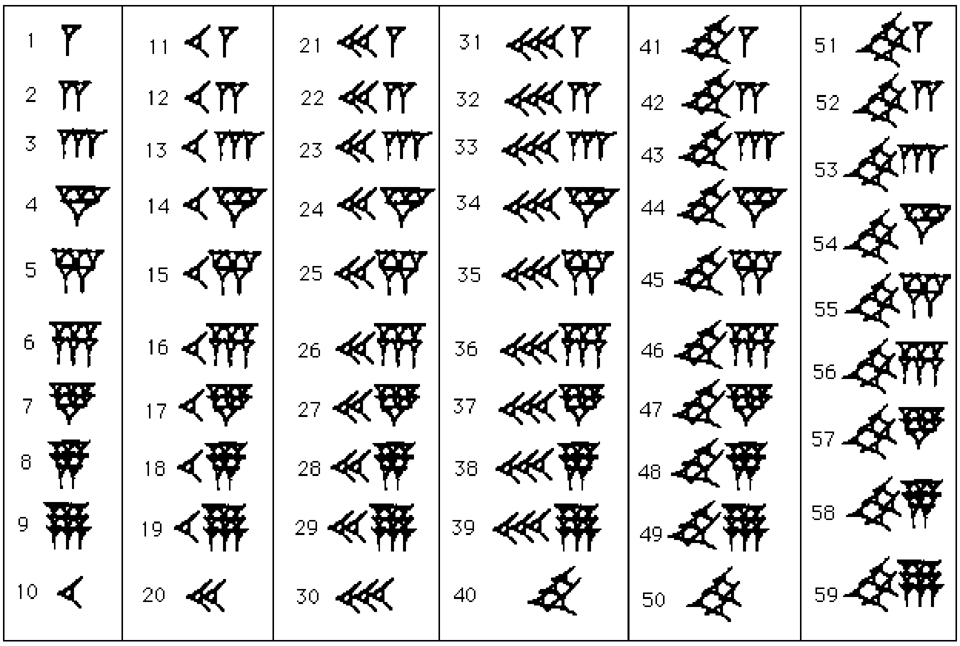
*Ancient Egypitian Hieroglyphs for Natural Numbers:*
*stroke, hobble for cattle, coil of rope, lotus plant, finger, frog, man*



*The Maya's numners*

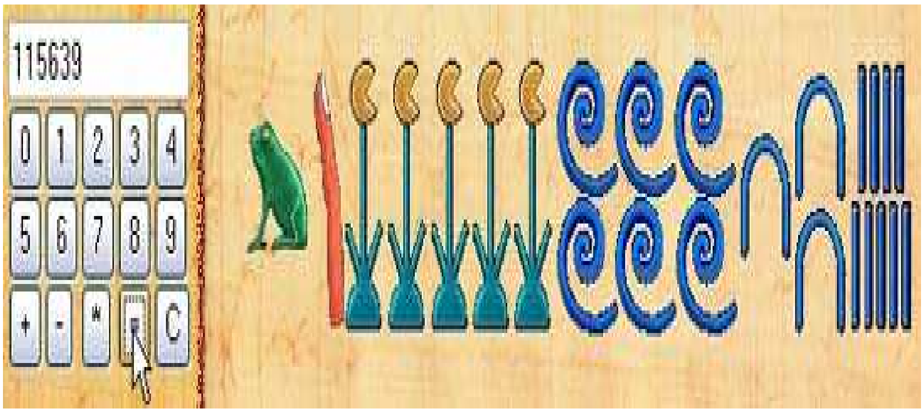


*The Babylonian numbers*



The Egyptyans had a decimal system using seven different symbols. The conventions for reading and writing numbers is simple: the higher numbers is always written in front of the lower number and where there is more than one row of numbers the reader should start at the top.

115639=100000+10000+5000+600+30+9




Among the Maya's numbers we can find a shell. Its meaning is the zero. The meaning of point • is one. The meaning of lying stroke --- is five. The point means more if it stands in higher position:

$$1 = \bullet$$

$$20 = \begin{cases} \bullet \\ (\;) \end{cases}$$

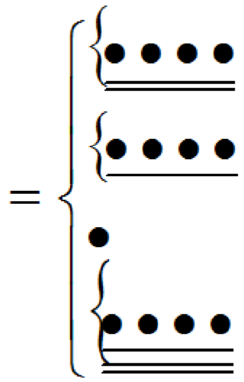
$$400 = \begin{cases} \bullet \\ (\;) \\ (\;) \end{cases} = 20^{2}$$

$$8000 = \begin{cases} \bullet \\ (\;) \\ (\;) \\ (\;) \end{cases} = 20^{3}$$

$$160000 = \begin{cases} \bullet \\ (\;) \\ (\;) \\ (\;) \\ (\;) \end{cases} = 20^{4}$$

*Expressing 115639 by Maya Numbers*

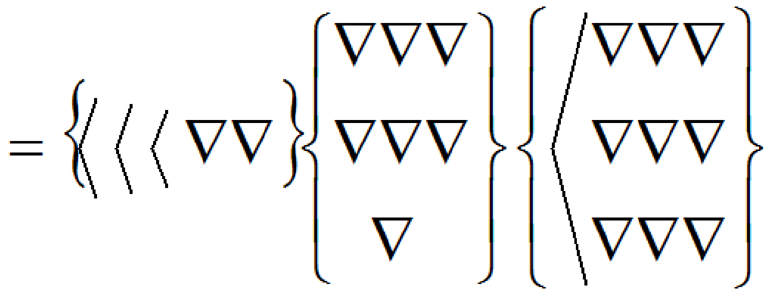
$$14 \cdot 20^3 + 9 \cdot 20^2 + 1 \cdot 20 + 19 =$$



In the system of babylonian numbers two symbols are used, merely. The meaning of the symbol $\nabla$ is one, but depending on the text-neighbourhood it may be sixty, too, as the babylonian system is a sexagesimal sytem with positional value. The meaning of the symbol $\langle$ is ten and the zero was denoted by an empty place. For example we express 115639 by the babylonian sytem

*Expressing 115639 by Babylonian Numbers*

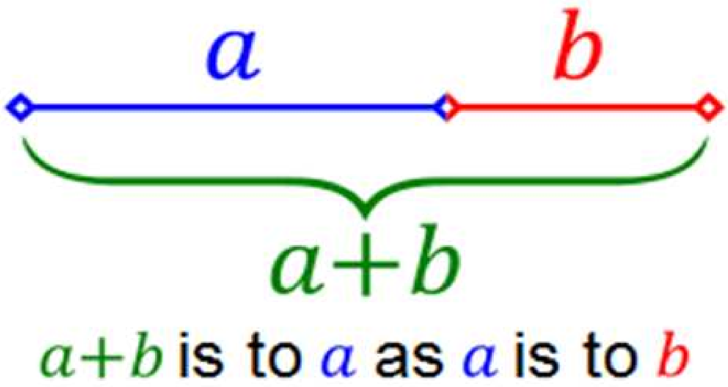
$$32 \cdot 60^2 + 7 \cdot 60 + 19 =$$



## Mathematics in Fine Arts

A common base of the classical architecture, sculpture and Renaissance painting is the golden section, where a passage is divided into two parts as follows.

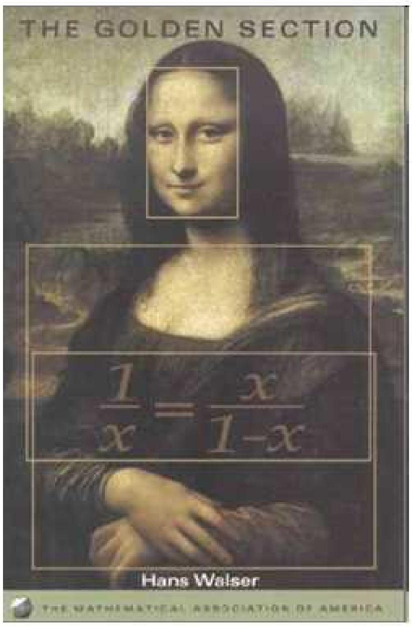
*Golden section*




$$(a+b):a = a:b$$

An application of golden section is demonstrated by the well known painting of Leonardo Da Vinci, „Mona Lisa”.

*Leonardo Da Vinci: Mona Lisa*




$$a+b=1$$
$$a=x\,,b=1-x$$
$$(a+b):a=\frac{1}{x}$$
$$a:b=\frac{x}{1-x}$$
$$x^2+x-1=0$$
$$x>0$$
$$x=\frac{-1+\sqrt{5}}{2}$$

The ratio $\frac{1+\sqrt{5}}{2} \approx 1{,}618$ of golden section is aesthetically pleasing. (See the position of eyes on the face of Mona Lisa.) For this reason it plays an important role in the modern photography, too. With respect to Mathematics, the number $\sqrt{5}$ is especially interesting, because it is an irrational number. Now, we prove this fact. Using the indirect way we assume, that $\sqrt{5}$ is a rational number, that is there exist relative prime natural numbers *m* and *n* such that

(*) $$\sqrt{5} = \frac{m}{n}.$$

Hence $5n^2 = m^2 \Rightarrow 5|m^2 \Rightarrow 5|m$. (Because 5 is a prime number.) This means, that

(**) $$m = 5p,$$

where *p* is a natural number. Moreover,

$$m = 5p \Rightarrow m^2 = 25p^2 \Rightarrow 5n^2 = 25p^2 \Rightarrow n^2 = 5p^2 \Rightarrow 5|n^2 \Rightarrow 5|n.$$

(Meantime, the equality $m^2 = 5n^2$ was used, again.) This means that

(***) $$n = 5q,$$

where *q* is a natural number. By (**) and (***) we can see that the natural numbers *m* and *n* are not relative primes, so we obtain a contradiction. Finally, we get that the equality (*) is false, which gives that $\sqrt{5}$ is an irrational number.

## Mathematics in Music

One of greatest achievements of the Pithagoreans is the construction of the sound scale. Let us consider a string (monochord) with a lenght of 12 units. Plucking the monochord we hear a sound called „c”. Its octave (upper „c”), quart („f”) and quint („g”) have the lengths 6, 9 and 8 units, respectively. Hence the ratios are for octave $6:12 = 1:2$, quart $9:12 = 3:4$ and quint $8:12 = 2:3$, respectively. The main idea of the Pithagorean's construction is the quint – jumping. The first quint - jumping is the quint of „g” having the lenght $8 \cdot \frac{2}{3} = \frac{16}{3}$ units. This is called upper „d”, denoted by d’. The lower octave of d’ with the lenght $\frac{16}{3} \cdot 2 = \frac{32}{3}$ units gives the sound „d”. The second quint – jumping is the quint of „d” is the sound „a” with the lenght $\frac{32}{3} \cdot \frac{2}{3} = \frac{64}{9}$ units. The third quint – jumping is the quint of „a” having the lenght of $\frac{64}{9} \cdot \frac{2}{3} = \frac{128}{27}$ units. This is called upper „e” denoted by e’. The lower octave e’ with the lenght $\frac{128}{27} \cdot 2 = \frac{256}{27}$ unit gives the sound „e”. The last quint – jumping is the quint of „e” is the sound „h” with the lengt of $\frac{256}{27} \cdot \frac{2}{3} = \frac{512}{81}$ units.

Let us make a sound scale!

Jumpings: octave 1:2, quart 3:4, quint 2:3

2:1

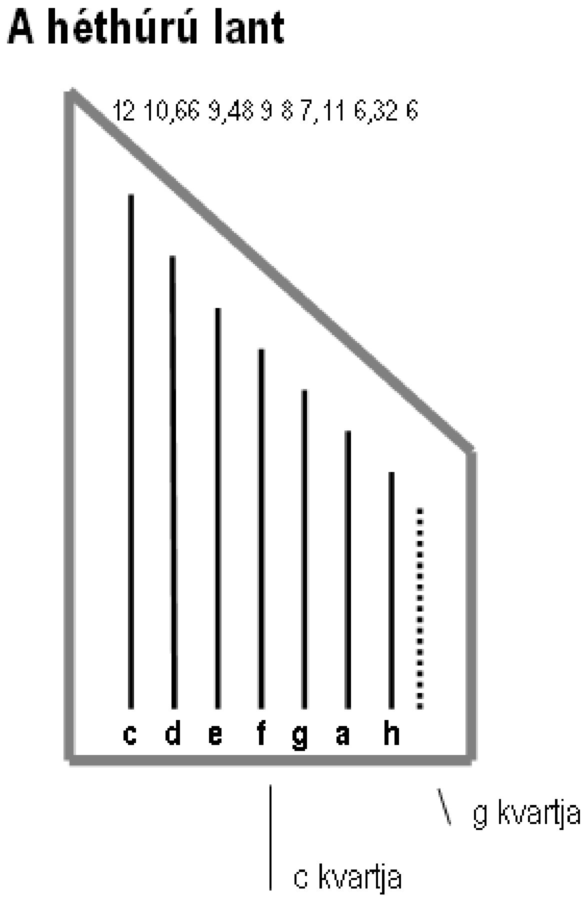
| | c | d | e | f | g | a | h | c' | d' | e' |
|---|---|---|---|---|---|---|---|---|---|---|
| 7. Jumping | | | | | | | | | | 128:27 |
| 4. Jumping | | | | | | | | | 16:3 | |
| 1. Jumping | | | | | | | | 6 | | |
| 9. Jumping | | | | | | | 512:81 | | | |
| 6. Jumping | | | | | | 64:9 | | | | |
| 3. Jumping | | | | | 8 | | | | | |
| 2. Jumping | | | | 9 | | | | | | |
| 8. Jumping | | | 256:27 | | | | | | | |
| 5. Jumping | | 32:3 | | | | | | | | |
| Start | 12 | | | | | | | | | |

We remark that the continuation of quint – jumping gives an error, because the quint of „h” has the lenght $\frac{512}{81}\cdot\frac{2}{3}=\frac{1024}{243}$ units and its lower octave has $\frac{1024}{243}\cdot 2=\frac{2048}{243}$ units. As $\frac{2048}{243}<9$ this sound is not the expected „f”. Ont he other hand the octave of „f” having the lenght $\frac{9}{2}$ units is greater than $\frac{2048}{243}$ units.

Finally, we have

| Sounds | Lenghts |
|---|---|
| c | 12 |
| d | $\frac{32}{3}\approx 10{,}67$ |
| e | $\frac{256}{27}\approx 9{,}48$ |
| f | 9 |
| g | 8 |
| a | $\frac{64}{9}\approx 7{,}11$ |
| h | $\frac{512}{81}\approx 6{,}32$ |
| upper c | 6 |

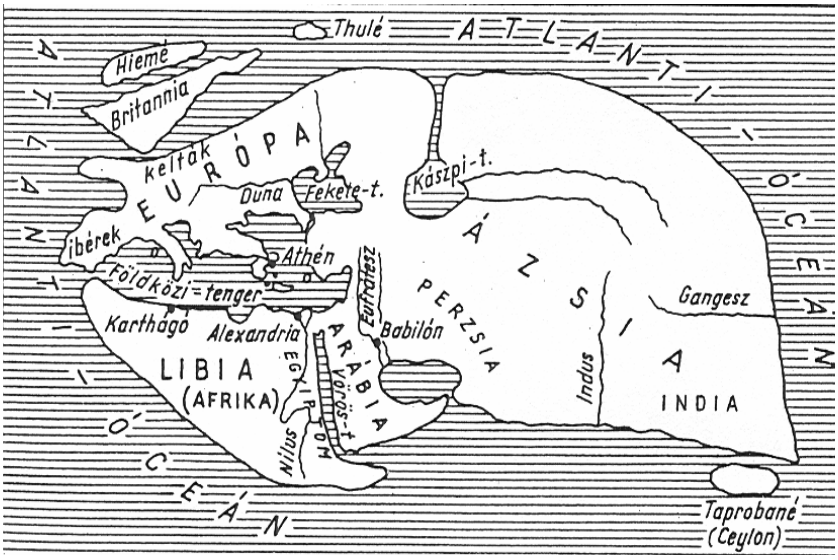
*The sound scale*




## Mathematics in Geography

Nowadays everybody knews that the circumference of Earth is about 40 000 km. The first man to calculate it was a Greek scientist (mathematician, geographer and astronomer) called Eratosthenes (c. 276 BC – c. 195 BC), the chief librarian of the Great Library of Alexandria. How did he do it? First of all, he had a map

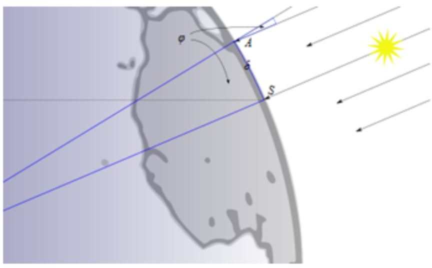
*The map of Eratosthenes*




We can observe that in the upper part a passage of Nile lies along an meridian which is a circle with the Earth's-centre in its centre and its radius is the Earth's- radius. Eratosthenes knew that on the summer solstice at local noon in Aswan (in the Ancient Times called Syene, situated on the Tropic of Cancer) the Sun would appear at the zenith, directly overhead. Moreover, he knew that in the Alexandria the angle of elevation of the Sun would be $7^\circ 12'$.

He estimated the distance between Alexandria and Aswan as 5000 stadia (about 900 km). (This estimation is interesting in itself: it took 50 days for a camel caravan to travel from Alexandria to Aswan with the speed $100\,\frac{\text{stadia}}{\text{day}}$.)

A sheet through Alexandria, Aswan and the Earth's centre



$$\frac{\text{Circumfer. of Earth}}{5000\,\text{stadia}} = \frac{360^\circ}{7^\circ 12'}$$

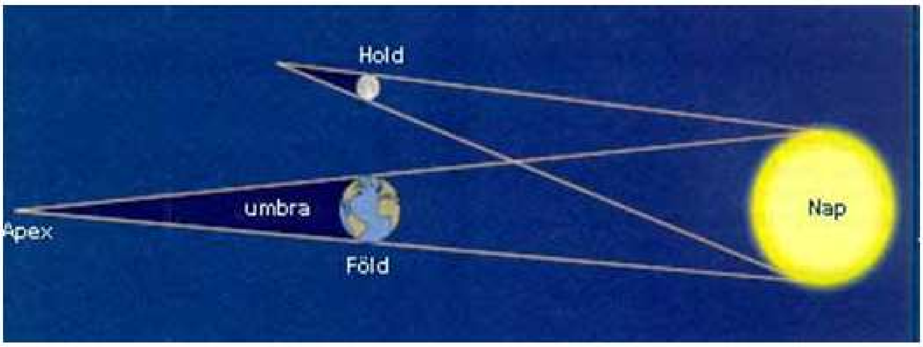
Hence, Circunference of Earth = 250000 stadia. The precision of this lenght is depending on the exact size of stadium. An *Attic stadium* was about 185 m, which would imply for Circumference of Earth about 46250 km.. In this case the relative error of measuring is

$\frac{46250-40000}{40000}=0{,}15625\approx 16\%$. If we assuming an „*Egyptian stadium*" with the lenght about 157,5 m the Circumference of Earth would be 39375 km. In this case the relative error of measuring is $\frac{40000-39375}{40000}=0{,}01625\approx 1{,}6\%$.

## Mathematics in Astronomy

Before Eratosthenes, an astronomer called Aristarchus (c. 310 BC – c. 230 BC), had already worked in Alexandria. He determined the relative sizes of the Earth and Moon. He observed that during a lunar eclipse the full Moon is passing through the shadow of the Earth:

$$\frac{\text{Time of entering of Moon in Earth's shadow}}{\text{Full time of Moon in Earth's shadow}}=$$
$$=\frac{\text{Diameter of Moon}}{\text{Diameter of Earth}}\Leftrightarrow \text{Moon'diam}=3476\text{km}$$




Eratosthenes having determined the Circumference of Earth (we use 40000 km) we know that diameter of the Earth = $\frac{40000}{\pi}\approx 12732$ km and using the ratio of Aristarchus

$$\frac{\text{Time of entering of Moon in Earth's shadow}}{\text{Full time of Moon in Earth's shadow}}=0{,}273$$

for Moon's diameter = 3476 km is obtained which is the absolute size of the Moon. Of course, Aristarhus did not know this result because in his time the diameter of the Earth was unknown. On the other hand his observation for the ratio

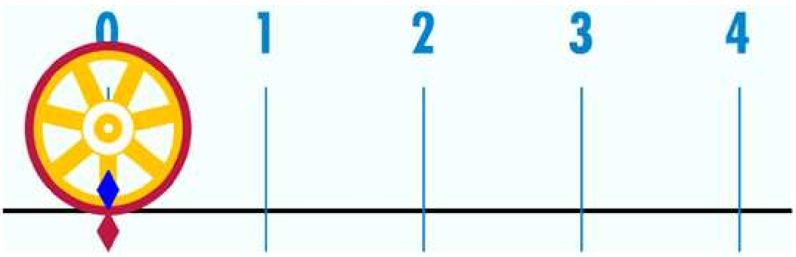
$$\frac{\text{Time of entering of Moon in Earth's shadow}}{\text{Full time of Moon in Earth's shadow}}=0{,}36$$

had a big relative error $\frac{0{,}36-0{,}273}{0{,}273}\approx 0{,}32=32\%$.

## Mathematics in itself

In the case of astronomy we can see the importance of the number $\pi = \frac{\text{Circle's circumfer.}}{\text{Circle's diameter}}$. It is known that the $\pi$ is an irrational number. In the next we demonstrate its estimation and approximation, too. For the estimation of $\pi$ we consider a circle with its diameter 1 unit in length.

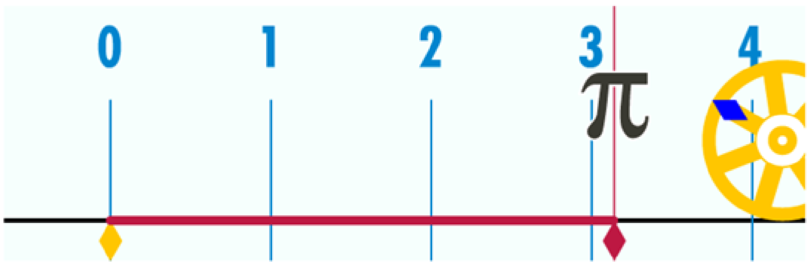
Estimation of $\pi$




Let us observe the movement of the signed point!

Estimation for Practice

$$\pi \approx 3.14$$




For the approximation of $\pi$ we use a calculator (CASIO *fx-570ES*). Our calculator says that

(1) $$\pi = 3{,}141592654$$

We know that is is false, because $\pi$ is an irrational number, so $\pi = \frac{3141592654}{1000000000}$ is impossibile. Then, either

(2) $$\pi < 3{,}141592654$$

or

(3) $$\pi > 3{,}141592654.$$

Which one is valid (2) or (3)? Our start - point is that the last digit „4” in (1) is a rounded digit by the calculator. Hence,

$$3{,}1415926535 < \pi < 3{,}1415926544 .$$

Clearly,

$$3{,}1415926535 < 3{,}141592654 < 3{,}1415926544$$

so, we have the approximation

(4) $$\left|\pi - 3{,}141592654\right| < 9 \cdot 10^{-10} < 10^{-9}.$$

Moreover,

$$314159265{,}35 < 10^{8} \cdot \pi < 314159265{,}44$$

so,

$$0{,}35 < 10^{8} \cdot \pi - 314159265 < 0{,}44 .$$

On the other hand our calculator says

$$10^{8} \cdot \pi - 314159265 = 0{,}35898 \;,$$

where on the right hand side the digit „8” is rounded, so

$$0{,}358975 < 10^{8} \cdot \pi - 314159265 < 0{,}358984 .$$

Hence,

$$314159265{,}358975 < 10^{8} \cdot \pi < 314159265{,}358984$$

and

$$3{,}1415926535\,8975 < \pi < 3{,}1415926535\,8984 .$$

As

$$3{,}1415926535\,8975 < 3{,}1415926535\,898 < 3{,}1415926535\,8984$$

we obtain the approximation

(5) $$\left|\pi - 3{,}1415926535\,898\right| < 9 \cdot 10^{14} < 10^{-13}.$$

Returning to our question, the approximation (4) is not sufficient for answer, but by approximation (5) we can write

$$\pi = (\pi - 3{,}1415926535\,898) + 3{,}1415926535\,898 \le$$
$$\le |\pi - 3{,}1415926535\,898| + 3{,}1415926535\,898 <$$
$$< 10^{-13} + 3{,}1415926535\,898 = 3{,}1415926535\,899 < 3{,}1415926534$$

that is the inequality (2) is true and inequality (3) is false.