EIGHT GRADERS' CAPABILITIES IN EXPONENTS: MAKING MENTAL COMPARISONS

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The general purpose of this study was to describe and analyze students' abilities in comparing exponents. The research was carried out at two different elementary schools of Aydın province in Turkey during 2009-2010 educational years. The study was conducted with 159 elementary school 8th grade students with the use of an exponent achievement test and in sample choice, random sampling method was used. General survey model was employed in order to determine students' characteristics in comparing exponents. Data collection tool, developed by researches with the help of a similar study, comprised of 20 open ended items. Items were developed by following the objectives of renewed elementary mathematics curriculum. When scoring the items, "1" is used for each correct answer and "0" is used for each incorrect answer or for items that were left empty. In each item, students were asked to compare a pair of exponents and choose the appropriate sign (>, =, <), without using a calculator. Students were also asked to write their reasoning when answering the items. Mathematics educators' views were taken into consideration to ensure the validity of the exponent achievement test. The results revealed that students were highly successful in computing exponents when base and power are used in natural number form. The study also showed that students had difficulties in comparing exponents especially when a decimal number is used as a base and a natural number used as a power. Finally, it was found that students' proficiency in comparing exponents varied when elements of different number sets used as a base and a power.

Keywords: Exponents, 8th graders, comparison levels, number sets

Although most students perceive exponents as a new number set, this concept in fact represents the abbreviated form of repeated multiplication. In math classes, teachers often give more emphasis on the procedural knowledge of this concept rather than teaching conceptually. In other words, teachers focus on examples that make children interiorize exponents as multiplying the base with itself as many as the power. Not having students sense the functional structure of exponents leads to some challenges and misconceptions. For instance, when computing the numerical value of an exponent, students often multiply base with the power and think that they arrived at a correct solution (Cengiz, 2006). Researches show that students most frequently have difficulty pointing exponential numbers in a number line. (Crider, 1998). This complexity may arise from not knowing the functional structure of exponents. Students who are not aware of the functional structure of an exponent consider it as two separate numbers. For instance, when solving the expression $\sqrt{4^9}$, students may handle 4^9 distinctly and operate as $\sqrt{4}=2, \sqrt{9}=3$ and may think that $\sqrt{4^9}$ equals to 2^3.

The students perceive exponents as challenging, unnecessary and complicated concepts and also they think that exponents have no connection with everyday life (Şenay, 2002). These mistakes generally, originates from the lack of exponential number sense and from overgeneralizing all rules that are true for natural numbers, integers and rational numbers to exponents and roots (Duatepe Paksu, 2008). Also, students have difficulty when they try to consider the relationship between procedural and structural conceptions of exponents (Kieran, 1992).

Using properties of natural numbers and integers for exponential numbers, indicates the importance of first learnt natural number set, in other words, it shows the importance of number prototypes in learning new concepts. Prototypes are the origins of new concepts and they are the first forms of an individual's ideas (Ülgen, 2001). Prototypes are the first examples constructed for a concept (Vinner & Hershkowitz, 1983). Prototypes have been found to be important in categorical or conceptual learning (Schwarz & Hershkowitz, 1999). Prototypical examples are used as "cognitive reference points" for the formation and judgement concerning membership of other examples in the category (Rosch & Mervis, 1975).

Researchers generally examined the common mistakes and difficulties of students on exponents (Cengiz, 2006; Crider, 1998; Orhun, 1998; Şenay, 2002). The difficulties occur for instance when students can't point exponents on number line due to not being able to determine the magnitude of a given exponent (Crider, 1998). Some students misinterpret the identity element of addition, zero, and think that the 0^{th} power of an exponent equals to the exponent itself (Cengiz, 2006; Crider, 1998).

Very few studies within mathematics education literature dealt with the development of teaching and learning exponents and little is known about the mental constructions students can make to develop a meaningful understanding of exponents or logarithms (Barnes, 2006; Chua, 2006). Three types of study have focused directly on students' understanding of exponents. The first one is about the perception of exponential growth, the second is about the development of intuitive understanding of exponential expressions and the third is about the development of teaching and learning exponents (Sastre & Mullet, 1998). Perception of exponential growth of a numerical series has been the subject of several studies (Wagenaar, 1982). These studies, even if not directly concerned with the actual problem of understanding of exponential expressions, show particularly convincingly that the majority of subjects, adults here, are not sensitive to the exponential nature of certain progressions (Sastre & Mullet, 1998). The second type of study has involved how students with a familiarity with exponentiation in expressions of the type a^n to estimate the magnitude of these expressions. Students were asked to depict an estimate of the magnitude of each expression graphically. Data analyses revealed that there were four different models of magnitude estimation: an exponential model, a multiplicative model, and two additive models (Mullet & Cheminat, 1995).

The work of Weber (2002) is a good example for the third type of study. Weber (2002) used Dubinsky's APOS theory (Dubinsky, 1991) in order to explain the way in which students develop their understanding of exponentiation as functions. Dubinsky (Dubinsky, 1991) extended Piaget's reflexive abstraction concept to more advanced topics going into undergraduate mathematics and beyond. His instructional model consists of four stages: action, process, object and schema. A student limited to an *action* understanding of exponents will be able to evaluate exponential functions only in the cases when the power is a given positive integer. Students with a *process* understanding of exponentiation can view exponentiation as a function and reason about properties of this function (e.g. 2^x will be a positive, increasing function). In the *process* stage of computing exponents, students' understanding of exponential functions only makes sense when their domain is restricted to the natural numbers (prototype concept). For exponential functions to become an *object* of thought, the student needs to be in a position to interpret situations where the number to be evaluated is not only a positive integer but a fraction, a negative number, or even an irrational number. For instance, to interpret $2^{1/2}$, the student must make sense of what "one half factor of 2" would be. Therefore, they should reason that $\sqrt{2}$ is a consequence of the definition of exponents and should also understand that the concept of power, originally defined only for the natural numbers, can be expanded to include zero, fractions, and real and complex numbers.

Although exponents are important mathematical concepts, little research has been done on students' learning and understanding of exponents. Researches in this topic, commonly includes the descriptions of instructional designs used in teaching exponents to students (Barnes, 2006; Weber, 2002). A number of researches have studied exponents to gain knowledge about secondary school students' mental constructions (Christou & Pitta-Pantazi & Zachariades, 2007; Mullet & Cheminat, 1995; Sastre & Mullet, 1998). In this sense, it is possible to say that researches concerning elementary school students' understanding of exponents are not sufficient enough.

For this reason, this study aims to determine elementary school 8^{th} graders achievement levels regarding mental computation of exponential expressions and to see whether different prototypes of exponents have impact on the mean scores of students.

Methods

Research Design

Survey method aims to collect data in order to determine certain features of a group (Büyüköztürk & Çakmak & Akgün & Karadeniz & Demirel 2009). With the help of general survey method singular or relational surveys can be made. Relational survey method aims to determine the covariation level between two or more variables (Karasar, 1999). This research is carried out by using relational survey method due to examining the role of prototypes in students' performances concerning mental comparisons of exponential expressions.

Sample

The sample of the study comprised of two different elementary schools in Aydın Province. 63 eighth graders (36 boys and 27 girls) from Beşeylül Elementary School and 96 eight graders (47 boys and 49 girls) from Turan Elementary School were chosen randomly. In total, 159 students took part in the study.

Instrument

"Exponential Numbers Achievement Test (ENAT)" was developed by researchers by considering items of a similar study (Christou, Pitta-Pantazi, & Zachariades, 2007) and used as a data collection tool in determining students' comparison skills. Items were developed in parallel with the targets of reviewed elementary school mathematics curriculum. ENAT consisted of 20 open ended items and was administered during a 30-minute session. ENAT comprised of 10 different prototypes and there were 2 items for each prototype. KR-20 method was employed to test for reliability since correct answers were scored as "1" and incorrect and empty answers were scored as "0" (Büyüköztürk et al., 2009), that is, items on the instrument were scored dichotomously, therefore Kuder-Richardson formula was used. Internal consistency of ENAT was computed by using KR-20 method and was found as 0, 74. Calculations were done via using EXCEL program. To increase the validity of the instrument, mathematics educators' and elementary school mathematics teachers' suggestions were followed.

In each item, students were asked to compare a pair of exponents and choose the appropriate sign ($>, =, <$), without using a calculator. Also, it was not possible for them to compute exponents by using paper and pencil, on the grounds that each item consisted of very large numbers. It was aimed that it would force students to use the properties of exponents instead of computing and to use their knowledge about number systems. Students were also asked to write their reasoning when answering the items.

ENAT consisted of two main groups: the group with same bases and the group with same powers. Items used in ENAT were shown in Table 1. given below.

Table 1. Exponents used in achievement test

Item No	Exponents with same powers	Item No	Exponents with same bases	Prototypes
1	$21^{11} \ldots 12^{11}$	**11**	$25^{20} \ldots 25^{24}$	$(N.N.)^{N.N.}$
2	$(0,4)^{-10} \ldots (0,4)^{-12}$	**10**	$(0,6)^{-10} \ldots (0,7)^{-10}$	$(P.D.N.)^{N.I.}$
3	$(0,2)^{13} \ldots (0,5)^{13}$	**19**	$(0,5)^{21} \ldots (0,5)^{17}$	$(P.D.N.)^{N.N.}$
4	$15^{-5} \ldots 16^{-5}$	**6**	$32^{-15} \ldots 32^{-19}$	$(N.N.)^{N.I.}$
5	$(-20)^{-10} \ldots (-20)^{-14}$	**15**	$(-8)^{-4} \ldots (-12)^{-4}$	$(N.I.)^{N.E.I.}$
7	$(-15)^{-13} \ldots (-15)^{-17}$	**20**	$(-9)^{-7} \ldots (-13)^{-7}$	$(N.I.)^{N.O.I.}$
8	$(-11)^{10} \ldots (-12)^{10}$	**16**	$(-16)^{8} \ldots (-16)^{10}$	$(N.I.)^{N.E.N.}$
12	$(-0,3)^{24} \ldots (-0,3)^{21}$	**17**	$(-0,1)^{15} \ldots (-0,2)^{15}$	$(N.D.N.)^{N.N.}$
9	$(-0,8)^{-20} \ldots (-0,8)^{-}$[illegible]	**13**	$(-0,3)^{-12} \ldots (-0,7)^{-1}$[illegible]	$(N.D.N.)^{N.I.}$
14	$(-12)^{7} \ldots (-12)^{13}$	**18**	$(-14)^{9} \ldots (-17)^{9}$	$(N.I.)^{N.O.N.}$

Curtailments:
1. N. N. : Natural Number *2. N. I. : Negative Integer*
3. P. D. N. : Positive Decimal Number, *4. N. D. N. : Negative Decimal Number*
5. N. E. I. : Negative Even Integer *6. N. E. N. : Natural Even Number*
7. N.O.I. : Negative Odd Integer *8. N.O. N. : Natural Odd Number*

Data Analyses

ENAT scores formed the bases of data analyses. After the necessary pre-processing, coding, and transferring to electronic storage, the results were analyzed by statistical package program (SPSS 15.0). To determine mean scores for each item, descriptive analysis was used. In addition, students' explanations for each item were examined in details in order to see which items were more challenging for them and the source of errors occurred in four items were diagnosed.

Results

The results are presented within the context of prototypes focusing on the understanding of exponents. Mean scores of student responses to ENAT items are presented in Table 2.

Table 2. Mean scores for each pairs of prototypes

Item No	Mean	Item No	Mean
1	0,95	**11**	0,92
2	0,33	**10**	0,60
3	0,87	**19**	0,24
4	0,69	**6**	0,84
5	0,69	**15**	0,69
7	0,28	**20**	0,38
8	0,61	**16**	0,74
9	0,79	**13**	0,63
12	0,77	**17**	0,70
14	0,72	**18**	0,86

When Table 2. is examined, it can be inferred that students were highly successful in computing exponents when base and power are elements of natural number set. Student performances were very high in Item 1 ($\overline{X} = 0,95$) and Item 11 ($\overline{X} = 0,92$) which are examples of $(natural\ number)^{natural\ number}$ prototypes. Students' high performance on these tasks can be explained by conventional examples given when teaching exponents. Teachers restrict themselves to using prototype concept of exponents which is defined as using elements of natural number set as a base and a power.

On the other hand, student performances were very low in Item 2 ($\overline{X}$ = 0,33), Item 7 ($\overline{X}$ = 0,28), Item 19 ($\overline{X}$ = 0,24) and Item 20 ($\overline{X}$ = 0,38). Responses given for these tasks were examined thoroughly and source of factors causing errors were identified. Crucial errors observed in Item 2, Item 7, Item 19 and Item 20 were explained respectively.

Item 2: $\mathbf{(0,4)^{-10} \ldots (0,4)^{-12}}$

In this task, two types of errors were observed. In the first case, students realize that these two exponents have same bases. However, they neglect bases and decide without considering base and power together. That is, they think that the one with bigger power must also be bigger than the other. So they operate as $-10 > -12$, $(0,4)^{-10} > (0,4)^{-12}$. In this task, bases are between 0 and 1. Although this affects the comparison, students are not aware of the mistakes they make. This can be explained by the effect of prototype extension to the other exponents that have elements from different number sets. Not having meaningful understanding of exponents leads to misconceptions. Children apply the rules of prototype concept to other exponents blindly and thus the process turns into algorithm and leads to errors.

In the second case, students also know that exponents have same bases. In addition, they know that $0 < 0,4 < 1$, and reason that the more you multiply 0,4 with itself, the smaller it is, however they neglect the effect of minus sign (-) existing in powers and operate as $10 < 12$ so $(0,4)^{-10} > (0,4)^{-12}$. Students are again under the effect of prototype extension. Because they make operations as if powers of exponents are natural numbers.

Item 7: $\mathbf{(-15)^{-13} \ldots (-15)^{-17}}$

In this task, two types of errors were observed. In the first case, although students know that the two exponents have negative integer bases and powers, they argue that -13> -17 so $(-15)^{-13}$ must be bigger than $(-15)^{-17}$. They make operations without considering that bases are negative integers, in other words; they neglect bases and follow the procedures used in prototype concepts.

In the second case, the students think that minus signs (-) in base and power cancel each other. More clearly, they carry out operations as $(-15)^{-13} = (-)(-).15^{13} = 15^{13}$, $(-15)^{-17} = (-)(-).15^{17} = 15^{17}$ and compare 15^{13} with 15^{17} and conclude that $(-15)^{-17}$ is bigger.

Item 19: $\mathbf{(0,5)^{21} \ldots (0,5)^{17}}$

The lowest success rate was monitored in this type of comparison; because most of the students paid no attention to the nature of numbers involved and followed rules based on generalizations. Also the influence of prototype view is more obvious since the elements of exponents consist of natural numbers. In this task, students directly compared powers without considering bases. They stated that the one with a bigger power will also be bigger and therefore they went on reasoning as 21 > 17, $(0,5)^{21} > (0,5)^{17}$.

Item 20: $\mathbf{(-9)^{-7}}$ **...** $\mathbf{(-13)^{-7}}$

When mean scores of students looked through it was found that students' success rates were low due to two possible reasons. Firstly, students processed with exponents without considering base and power together. In this task, students neglected powers as they did in other tasks. Then they reached a solution only by comparing bases. They knew that -9 > -13 so they argued $(-9)^{-7} > (-13)^{-7}$ must also be true. Secondly, students were aware of finding the multiplicative reciprocal of the base in order to have a positive value as a power. So they were successful when performing these steps: $(-9)^{-7} = \left(-\frac{1}{9}\right)^{7}$, $(-13)^{7} = \left(-\frac{1}{13}\right)^{7}$. However, they neglected the effect of minus sign. They thought that $(-9)^{-7} > (-13)^{-7}$ must be true, since $\frac{1}{9} > \frac{1}{13}$.

In order to search for different levels in ability to compare exponents mentally, data was analyzed by defining groups of students. Calculations were done on the basis of normal distribution. Mean score and standard deviation of the test was found to determine groups according to success rates ($\overline{X}$ = 13.29, SD = 2.992). Students that have total scores within $13,29 \mp 2.992$ that is, achievement scores within the interval (16.282, 10.298) were accepted as average achievers ($n = 105$). Students that have achievement scores higher than 16.282 were accepted as high achievers ($n = 27$). Finally students with achievements scores lower than 10.298 were accepted as low achievers ($n = 27$). Students' mean scores for each prototype are presented in Table 3.

Table 3. Mean scores for each prototype according to achievement levels

Prototypes	**Items**	**Mean for low achievers**	**Mean for average achievers**	**Mean for high achievers**
$(N.N.)^{N.N.}$	1-11	0,80	0,95	1,00
$(N.I.)^{N.O.N.}$	14-18	0,48	0,81	1,00
$(N.N.)^{N.I.}$	4-6	0,35	0,81	1,00
$(N.I.)^{N.E.N.}$	8-16	0,56	0,62	1,00
$(N.I.)^{N.E.I.}$	5-15	0,39	0,71	0,91
$(N.D.N.)^{N.N.}$	12-17	0,45	0,76	0,87
$(N.D.N.)^{N.I.}$	9-13	0,28	0,78	0,86
$(P.D.N.)^{N.I.}$	2-10	0,21	0,45	0,82
$(P.D.N.)^{N.N.}$	3-19	0,48	0,51	0,80
$(N.I.)^{N.O.I.}$	7-20	0,18	0,50	0,74

When Table 3. is examined, it is clear that all high achievers were able to compare Item 1, Item 4, Item 6, Item 8, Item 11, Item 14, Item 16 and Item 18 correctly. Students in all the groups successfully computed Item 1 and Item 11 which involved exponents with bases and powers natural numbers. These items reflect the action element of Dubinsky's APOS theory (1991). Students have high performances when exponents have elements from natural numbers since teachers initially give examples in this type during instruction. As Schwarz and Hershkowitz (1999) stated, exponents with bases and powers natural numbers serve as "cognitive reference points" for the computation of other exponents. All groups exhibited lower achievement in Item 2, Item 3, Item 7, Item 9, Item 10, Item 13, Item 19 and Item 20. These items contain elements from natural numbers, integers and decimal numbers. This show that students often neglect bases or powers when making comparisons and this way of thinking has leads them to errors. These errors arise from the influence of prototype concept of exponents. In other words, students try to apply the properties of exponents with natural number base and exponent to every type of exponent that have elements from different number sets. In general, students face with challenges if they don't have a deep understanding of exponent concept since most of the questions administered require students to use their conceptual knowledge.

Discussion and Conclusions

Based on the results of the study students in three groups frequently followed the procedural understanding of exponent concept. That is, students tried to use repeated multiplication procedure to every task they administered. Students are introduced with this definition quite early at schools. However, Schwarz and Hershkowitz (1999) state that prototype concept of exponents can be detrimental to conceptual learning. In addition, researchers have pointed out (e.g, Confrey & Smith, 1995; Lakoff & Nunez, 2000) that representation of exponents as repeated multiplication is inadequate to perform much of the reasoning. Expressions such as 2^{-1} and $2^{1/2}$ will make no sense for a student who can only view exponents as repeated multiplication because multiplying a number by itself negative one or half times will not be possible. Contrary to process stage of exponents, Breidenbach, Dubinsky, Hawks and Nichols (1992) stated that understanding exponentiation as a function first requires understanding this concept as a process. In addition he study of Weber (2002) revealed that students' understanding of exponents and logarithms was rather limited and most students were incapable of understanding exponents and logarithms as processes. In the present study, 8^{th} graders exhibited difficulties when comparing exponential expressions. The present data are congruent with the results of Sastre and Mullet (1998). Besides, students employed an additive model when they were asked to compare exponential expressions. (Additive model employers think that the higher the base, the higher the value of expression and the higher the power the higher the value of expression). So, the data are also congruent with the results reported by Wilkening and Anderson (1991), in which the participants were asked to estimate the volume of cones of different sizes. Adults, adolescents and children employed exponential, multiplicative and additive model respectively. Mullet and Cheminat (1995) investigated how students with a familiarity with exponentiation intuitively combined information about bases and powers in expressions of the type a^n to estimate the magnitude of these expressions. They found in their study that dominant patterns of estimation were additive and multiplicative. These results are in line with the findings of the present study. Baxter and Dole (1990) stated that few errors were random or careless and many errors were in fact conceptual and learned and have become habitual and consistent with advancing years in school. These findings are in agreement with the current study.

Attributing learners' mistakes to low intelligence, low mathematical aptitude, perceptual difficulties or learning disabilities is not useful in rectifying mistakes. These factors naturally play a role but if it is our intention to help the individual learner, we need to examine available detail and determine the specific roots of the mistakes (Olivier, 1989). Instead of being originally text book and law bound in teaching, new instruction methods that foster structural understanding in exponents should be adopted. Further research on students' understanding of exponents and logarithms is needed since the importance of these functions and the low level of understanding justify the call for research. Also, an investigation of how students understand the symbols and the notation associated with exponents can be searched in order for students to overcome challenges with exponential expressions.

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